MULTISTAGE GAME MODELS AND DELAY SUPERGAMES

Nobel Lecture, December 9, 1994

by

REINHARD SELTEN, BONN

Rheinische Friedrich-Wilhelms-Universität, Bonn, Germany

ABSTRACT

The order of stages in a multistage game is often interpreted by looking at earlier stages as involving more long term decisions. For the purpose of making this interpretation precise, the notion of a delay supergame of a bounded multistage game is introduced. A multistage game is bounded if the length of play has an upper bound. A delay supergame is played over many periods. Decisions on all stages are made simultaneously, but with different delays until they become effective. The earlier the stage the longer the delay.

A subgame perfect equilibrium of a bounded multistage game generates a subgame perfect equilibrium in every one of its delay supergames. This is the first main conclusion of the paper. A subgame perfect equilibrium set is a set of subgame perfect equilibria all of which yield the same payoffs, not only in the game as a whole, but also in each of its subgames. The second main conclusion concerns multistage games with a unique subgame perfect equilibrium set and their delay supergames which are bounded in the sense that the number of periods is finite. If a bounded multistage game has a unique subgame perfect equilibrium set, then the same is true for every one of its bounded delay supergames.

Finally the descriptive relevance of multistage game models and their subgame perfect equilibria is discussed in the light of the results obtained.

1. Introduction

In the economic literature one finds many game models, in which the players make simultaneous decisions on each of a number of successive stages, always fully informed about what has been done on previous stages. An early example can be found in my paper " a simple model of imperfect competition where 4 are few and six are many (Selten, 1973). In this oligopoly model the players first decide whether they want to take part in cartel bargaining. This participation stage is followed by a cartel bargaining in which quota cartels can be proposed and agreed upon. The third and last stage is a supply decision stage in which production quantities are fixed.

Another example is a two stage duopoly model with production capacity decisions on the first stage and Bertrand price competition on the second stage (Kreps and Scheinkman, 1983). Multistage game models can be analyzed on the basis of the subgame perfect equilibrium concept (Selten, 1965), the simplest refinement of ordinary game theoretic equilibrium (Nash, 1951). Sometimes additional selection criteria are combined with subgame perfect equilibria, like symmetry and local efficiency in the case of my above mentioned model. The analysis of my model yields the result that up to 4 competitors always form a cartel whereas in the presence of 6 or more competitors the probability of cartel formation is less than 2%. The model by Kreps and Scheinkman deepens our understanding of Cournot's (1838) oligopoly theory. In equilibrium Cournot quantity competition takes place on the first stage of their model. The examples show that the analysis of multistage game models can lead to interesting theoretical conclusions.

In some cases it may be justified to look at the stages of a multistage game model as decision points succeeding each other in time. However, often this direct temporal interpretation is not adequate. Many multistage game models do not really aim at the description of situations in which decisions must be made in a fixed time order. Thus the model of Kreps and Scheinkman places capacity choice before price choice, not because price choice cannot precede capacity choice, but rather because capacity decisions are considered to be more long term than price decisions. It seems to be natural to consider long term decisions as fixed, when short term decisions are made. "Term length" in the sense of a position on a short term-long term scale rather than time is the consideration underlying the order of stages. If decisions of greater term length are modelled as made on an earlier stage, this is intended to have the effect that in subgame perfect equilibrium more short term decisions are based on more long term decisions taken as fixed.

The term length interpretation looks at a multistage game model as a condensed description of an ongoing situation in which stage decisions are not made once and for all. Strategic variables may change over time, but more short term decisions are in some way subordinated to more long term ones. The reasons for this subordination are not explicitly spelled out.

Simple oligopoly models like the Cournot model which involve only one stage are usually also interpreted as condensed descriptions of ongoing situations. The literal interpretation as a one-shot game would leave little room for applied significance. One must think of such models as being played repeatedly in a supergame (Aumann, 1959, Friedman, 1977), but without making use of the potential for quasicooperation which may be present in such situations. The analysis of the one shot game instead of the supergame amounts to the assumption that any kind of collusion is excluded by effectively enforced cartel laws or for other reasons.

Similarly one could look at a multistage game model as representing the structure of a period in a supergame. However, this would mean that one sticks to the direct temporal interpretation of the order of stages. Obviously the term length interpretation requires a different picture of the underlying ongoing situation. It is the modest purpose of this lecture to provide an explicit model of the underlying situation which justifies the reduction to the multistage game with its subordination of the shorter term to the longer term.

What does it mean in the model by Kreps and Scheinkman that capacity decisions are long term and price decisions are short term? It could mean that due to exogenous institutional circumstances capacities can be adjusted only at certain points in time, say at the beginning of the year, whereas prices can be changed every day. Alternatively one could think of a difference of change costs, high for capacity adjustments and low for price changes. This would lead to a model in the spirit of the inertia supergame (Marschak and Selten, 1978). A third possibility is the answer given here. It focusses on the delay between the time a decision is made and the time at which it becomes effective. This delay may be two years for capacity adjustments and only one day for price adjustments. The longer the delay the more long term the decision is considered to be.

It seems to me that in many cases the difference between a more long term and a more short term decision is adequately explained as a difference of delay. Of course, in some multistage game models necessities of temporal order, exogenous institutional circumstances and differences of change costs may also be considerations in the interpretation of the order of stages. However, here we shall only be concerned with the term length interpretation elaborated by looking at differences of term length as differences of delay times needed until a decision becomes effective. The underlying ongoing situation will be modelled as a special kind of game, called a "delay supergame". In a delay supergame decisions on all strategic variables are made at the same time, period after period, but these decisions become effective with different delays. Thus, in period t decisions on the price in t+l and capacity in t+10 may be made, on both variables at the same time and simultaneously by all players.

In a delay supergame the players have full information about previous history of play, but not about simultaneous decisions made by other players. All decisions made in a period become publicly known at the beginning of the next period.

It does not really matter exactly how long the delays are. For the analysis of delay supergames only the order of the decision variables with respect to delay length matters.

A delay supergame is not necessarily played for a fixed number of periods; the definition will involve a probability distribution over the number of periods played. We speak of a "bounded" delay supergame if the number of periods has a finite upper bound and of an "unbounded" one otherwise. The distinction between bounded and unbounded delay supergames is game theoretically important.

Every subgame perfect equilibrium of a bounded multistage game always generates a subgame perfect equilibrium for every one of its bounded or unbounded delay supergames. This is the first main conclusion of the paper (theorem 1 in 5.3). The generated equilibrium can roughly be described as the repeated application of the multistage game equilibrium strategies in every period played.

In general, a delay supergame and especially an unbounded one may have many additional subgame perfect equilibria. It is well known that this happens already in ordinary supergames (Rubinstein, 1976, 1980, Benoit and Krishna, 1985). Since normal form games are special multistage games with only one stage, supergames are special delay supergames.

A subgame perfect equilibrium set is a set of subgame perfect equilibria all of which yield the same payoffs not only in the game as a whole but also in each of its subgames. A multistage game or a delay supergame will be called "determinate" if the set of all of its subgame perfect equilibria is a subgame perfect equilibrium set. Every bounded delay supergame of a determinate bounded multistage game is determinate. This is the second main conclusion of this paper (theorem 2 in 5.5).

This lecture will not be concerned with the question which kind of "folk theorems" hold for which class of delay supergames. Such theorems are interesting from the point of view of normative game theory, but their applied significance is limited. Finite supergames of prisoners' dilemma games have only one subgame perfect equilibrium which prescribes the non-cooperative choice everywhere, but nevertheless experienced experimental subjects cooperate in such games until shortly before the end (Selten and Stoecker, 1986). On the other hand in some supergame-like oligopoly experiments cooperation is not observed (Sauermann and Selten, 1959, Hoggatt, 1959, Fouraker and Siegel, 1963, Stern, 1967). It is an empirical question under which conditions behavior in a delay supergame converges to a subgame perfect equilibrium of the underlying multistage game. At the end of the paper this problem will be discussed in more detail.

Instead of the usual framework of the extensive game (von Neumann and Morgenstern, 1944, Kuhn, 1953, Selten, 1975) a somewhat different one is used here, which is especially adapted to multistage games and their delay supergames. Simultaneous decisions are represented as being made at the same history of previous play and information is not explicitly modelled. A "choice set function" defined recursively together with a "path set" take over the role of the game tree. As in the usual extensive form a "probability assignment" describes the probability of random choices and a payoff function determines the payoffs at the end of a play. The framework could be made as general as that of an extensive game by the additional introduction of information partitions for the players but this will not be done here.

Even if our main conclusions are intuitively plausible and not surprising some formalism is necessary to make statements and their proofs precise.

2. Multistage games

A multistage game will be defined as a structure built up of four constituents, a start s, a choice set function A, a probability assignment p, and a payoff function h. The start s is a symbol which represents the situation before the beginning of the game. The choice set function describes what choices are available to the players in every situation which may arise in the game. The probability assignment p assigns probabilities to random choices and the payoff function h specifies the payoffs at the end of the game. Detailed formal definitions are given below.

2.1 The choice set function

A multistage game involves n personal players 1,...,n and a *random player 0* (interpreted as a random mechanism). In the following we present a joint recursive definition of a *choice set function* and the notion of a *path* (a path represents a previous history of play). In addition to this, further auxiliary definitions like that of a play, a preplay, and a choice combination are introduced. Interpretations are added in brackets.

1. The *start* s is a *path.*
2. If u is a path then the *choice set function* A assigns a *choice set* $A_i(u)$ to every player i = 0,...,n.

Auxiliary definitions and notations: A player i is called *active* at a path u if $A_i(u)$ is non-empty and *passive* otherwise (i = 0,...,n). A path u is a *play* if all players i = 0,...,n are passive at u and a *preplay* otherwise. (A play represents a history from the beginning to the end; a preplay still has to be continued.) The set of all active players at a path u is denoted by N(u). A *choice combination* at a preplay u is a system

$$a = (a_i)_{i \in N(u)} \text{ with } a_i \in A_i(u) \text{ for all } i \in N(u)$$

The set of all choice combinations at u is denoted by A(u).

3. If u is a preplay and a is a choice combination at u, then v = ua is a *path.* All paths are generated in this way.

Notation: The set of all paths is denoted by U, the set of all plays by Z and the set of all preplays by P . According to 3. a path u = s $a^1 \ldots a^k$ is built up as a sequence beginning with the start s and continued by successive choice combinations $a^1,\ldots,a^k$.

Finiteness of random choice sets: We only consider choice set functions with the additional property that all random choice sets $A_0(u)$ are finite. In this way we avoid tedious technicalities. In the following finiteness of all random choice sets will always be assumed.

2.2. The probability assignment

A probability assignment p assigns a probability distribution p_u, over $A_0(u)$ to every preplay at which the random player is active.

Auxiliary definitions and notation: The probability assigned to a choice $a_0 \in A_0(u)$ by p_u is denoted $p_u(a_0)$. The *length* of a path $u = sa^1...a^k$ is the number k of choice combinations following s in u. The length of u is denoted by $|u|$.

Comment: Our framework does not exclude multistage games without an upper bound on the length of a preplay. Multistage game models usually have a finite number of stages. Arbitrarily long preplays cannot arise in such models. However, we aim at a definition which also covers delay supergames without any bound on the number of periods. Unbounded delay supergames will involve stopping probabilities which have the effect that with probability 1 the game ends in finite time and that expected payoffs can be defined. In order to achieve this purpose for our general framework we impose a joint condition on the choice set function A and the probability assignment p.

Random stopping condition: A positive integer μ and a real number δ with $0 < \delta < 1$ exist such that for every preplay u with $|u| \geq \mu$ the choice set $A_0(u)$ contains a choice ω) with $p_u(\omega) \geq \delta$ and with the following property: If ω is the random player's component of $a \in A(u)$, then v = ua is a play.

We shall only consider multistage games, for which the random stopping condition is satisfied. It will always be assumed that this is the case.

2.3 The payoff function

A *payoff function* h is a function which assigns a *payoff vector*

$$h(z) = (h_1(z),..., h_n(z))$$

to every play $z \in Z$. The components $h_i(z)$ of h(z) are real numbers. $h_i(z)$ is *player i's payoff* for z.

Boundedness of payoffs: We only consider payoff functions with the property that constants C_0 and C_1 exist such that

$$|h_i(z)| \leq C_0 + C_1 |z|$$

holds for every play $z \in Z$ and for i = 1,...,n.

We impose this boundedness condition in order to make sure that expected payoffs can be defined. In view of the intended application to delay supergames, it is important to permit an increasing linear dependence on the length of a play. It will always be assumed that payoffs are bounded in this way.

2.4 *Definition of a multistage game*

A *multistage game*
$G = (s, A, p, h)$
is composed of four constituents, a start s, a choice set function A, a probability assignment p and a payoff function h, with the properties explained above (see 2.1, 2.2, and 2.3).

A multistage game G = (s, A, p, h) is called *bounded,* if in G the length of a path is bounded from above. Obviously in such games a maximum length M of a path exists. It can be seen immediately that the existence of this maximum length M implies that the random stopping condition of 2.2 holds with $\mu = M$ simply because there are no preplays u with $|u| \geq M$.

2.5 **Strategies**

In sections 2.5 - 2.7 all definitions will refer to a fixed but arbitrary multistage game G = (s, A, p, h). The set of all preplays at which player i is active is denoted by P_i. We call P_i *player i's preplay set.* A *local strategy* of a personal player i at a preplay $u \in P_i$ is a probability distribution b_{iu} over player i's choice set $A_i(u)$ which assigns positive probabilities to finitely many choices only. The probability assigned to a choice $a_i \in A_i(u)$ by b_{iu} is denoted by $b_{iu}(a_i)$.

A *behavior strategy* b_i of player i is a system of local strategies

$$b_i = (b_{iu})_{u \in P_i}$$

specifying a local strategy b_{iu} for every preplay u of player i. Player i's preplay set P_i may be empty. In this case the definition of a behavior strategy is to be understood in such a way that player i has exactly one behavior strategy, the *empty* strategy.

Comment: In multistage games every player is always fully informed about all choices on previous stages. This implies that such games have perfect recall. It is clear that the extensive games representing a multistage game have the formal property of perfect recall as it is usually expressed (Kuhn 1953, Selten 1975). Kuhn (1953) has proved a theorem which shows that without any essential loss the noncooperative analysis of finite extensive games with perfect recall can be restricted to behavior strategies. Aumann (1964) has generalized this theorem to extensive games in which a continuum of choices may be available in some choice sets. This is important in the context of multistage game models which usually involve continuously varying decision parameters.

In order to avoid tedious technical detail we shall restrict our attention to finite local strategies, i.e. local strategies with a finite carrier and finite behavior strategies which specify such local strategies only.

Further definitions: A local strategy is called *pure* if it assigns probability 1 to one of the choices. Pure local strategies can be identified with choices. The set of all finite local strategies of a player i at a preplay $u \in P_i$ is denoted by B_{iu}. The set of all behavior strategies of a personal player i is denoted by B_i. A *pure strategy* of a personal player i assigns a pure local strategy or, in other words, a choice at u to every $u \in P_i$.

A *strategy combination* $b = (b_1,...,b_n)$ is an n-tuple specifying a behavior strategy b_i for every personal player. A strategy combination is called *pure,* if all its components are pure. The set of all strategy combinations is denoted by B.

2.6 Realization probabilities

Consider a strategy combination $b = (b_1,..., b_n)$ and a preplay u. For every personal player i active at u let b_{iu} be the local strategy assigned to u by b_i. For every choice combination

$$a = (a_i)_{i \in N(u)} \text{ with } a_i \in A_i(u)$$

we define the *conditional realization probability* of a at u as the product of $p_0(a_0)$ and all $b_{iu}(a_i)$ with i active at u. This probability is denoted by $b_u(a)$:

$$b_u(a) = p_u(a_0) \prod_{i \in N(u)-\{0\}} b_{iu}(a_i)$$

In this way a probability distribution b_u over A(u) is associated to a strategy combination $b = (b_1,..., b_n)$ and a preplay u.

Now consider a path $v \in V$ with $v = sa^1...a^k$. We say that a path $u = sa^1...a^j$ is *on* v if we have $j \leq k$ and the first choice combinations $a^1,...,a^j$ are the same in u and v. The *realization probability of v under* $b = (b_1,...,b_n)$ is the product of all $b_u(a)$ with ua on v: This probability is denoted by b(v).

$$b(v) = \prod_{ua \text{ on } v} b_u(a)$$

The realization probability of $v = sa^1...a^k$ is interpreted as the probability that in the course of playing the game with the strategies in b the play passes the choice combinations $a^1,...,a^k$ one after the other.

2.7 Expected payoffs

For every strategy combination $b = (b_1,...,b_n)$ we shall define *expected payoffs* $H_i(b)$ for every personal player i = 1,...,n. We shall focus on a fixed but arbitrary personal player i. In the case of a bounded multistage game G player i's expected payoffs are defined as follows:

$$H_i(b) = \sum_{z \in Z} b(z)h(z)$$

In the following we shall assume that G is an unbounded multistage game. In this case the definition of expected payoffs is essentially the same as in the bounded case, but it needs to be elaborated, since infinite sums do not necessarily converge. For k = 0,1,... let Z_k, be the set of all plays z with $|z| = k$. Player i's expected payoff $H_i(b)$ is defined as

$$H_i(b) = \lim_{T\to\infty} \sum_{k=0}^{T} \sum_{z\in Z} b(z)h(z)$$

It has to be shown that the limit exists. For k = 0, 1,... let Q_k be the set of all preplays v with $|v| = k$. Define

$$b(Z_k) = \sum_{z\in Z_k} b(z)$$

$$b(Q_k) = \sum_{v\in Q_k} b(v)$$

$$Y_k = \sum_{z\in Z_k} b(z)h_i(z)$$

Since all local strategies assign positive probabilities to finitely many choices only, these sums are finite. Let μ and δ be numbers such that the random stopping condition holds with these numbers. The random stopping condition permits the conclusion that for $k = \mu, \mu+1, \ldots$ we have

$$b(Q_{k+1}) \leq (1-\delta)b(Q_k)$$

and therefore

$$b(Q_k) \leq (1-\delta)^{k-\mu}b(Q_\mu)$$

Another conclusion from the second last inequality is the following one

$$b(Q_k) - b(Q_{k+1}) \leq \delta\, b(Q_k)$$

for $k = \mu, \mu+1, \ldots$ On the left hand side of the last inequality we find nothing else than $b(Z_{k+1})$ since after the next choice combination a preplay in Q_k becomes a preplay in Q_{k+1} or a play in Z_{k+1}. In view of $b(Q_\mu) \leq 1$ the last two inequalities permit the following conclusion:

$$b(Z_{k+1}) \leq \delta\,(1-\delta)^{k-\mu}$$

for $k = \mu, \mu+1, \ldots$. The boundedness condition for payoffs can now be used to bound Y_k:

$$|Y_k| \leq \delta(1-\delta)^{k-\mu-1}(C_0+C_1k) \quad \text{for } k = \mu, \mu+1, \mu+2, \ldots$$

where C_0 and C_1 are numbers with the properties required by the boundedness condition on payoffs. It can be seen without difficulty that the infinite sum of the terms on the right-hand side converges. Therefore the same is true for the terms on the left-hand side. This has the consequence that the limit exists, by which $H_i(b)$ is defined. We call $H(b) = (H_1(b),...,H_n(b))$ the *expected payoff vector* of b. The function H which assigns H(b) to every $b \in B$ is referred to as the *expected payoff function.*

3. *Equilibria*

In this section we shall first define equilibrium in the framework of the multistage game. Then we look at subgames and subgame perfectness will be defined. As before, all definitions refer to a fixed but arbitrary multistage game G = (s, A, p, h).

3.1 Equilibrium

An *i-incomplete strategy combination* b_{-i} is an (n-1)-tupel of behavior strategies b_j with one strategy for all personal players except player i:

$$b_{-i} = (b_{1},..., b_{i-1}, -, b_{i+1},..., b_n)$$

We use the notation b_ib_{-i} for the strategy combination b which contains b_i and the components b_j of b_{-i}. We say that a behavior strategy $\tilde{b}_i$ is a *best reply* to b_{-i}, if we have

$$H_i(\tilde{b}_ib_{-i}) = \max_{b_i \in B_i} H_i(b_ib_{-i})$$

$\tilde{b}_i$ is a *best reply* to $b = (b_1,...,b_n)$, if it is a best reply to the i-incomplete strategy combination b_{-i} formed by the components of b except b_i. We say that $\tilde{b} = (\tilde{b}_1,...,\tilde{b}_n)$ is a *best reply* to $b = (b_1,... b_n)$, if for i = 1,...,n the behavior strategy $\tilde{b}_i$ is a best reply to b.

A strategy combination $b = (b_1,... b_n)$ is an *equilibrium,* if it is a best reply to itself. *An equilibrium set* E is a set of equilibria with the property that the payoff vector H(b) is the same one for all $b \in B$.

3.2 Subgames

For every preplay u we define a *subgame* G^u at u of G. This subgame is the multistage game

$$G^u = (u, A^u, p^u, h^u)$$

whose constituents will be described in the following. The start is the preplay u and A^u is the restriction of A to the paths v such that u is on v. A path v of this kind permits a representation of the form:

$$v = ua^1...a^k$$

The set of all these paths is denoted by U^u. Paths in U^u are at the same time paths in G starting with s and paths in G^u starting with u. The probability assignment p^u is the restriction of p^u to U^u. The set of all plays in G^u is denoted by Z^u. The payoff function h^u is the restriction of h to Z^u. It can be seen immediately that u, A^u, p^u and h^u form a multistage game with all the properties required in 2.1, 2.2, and 2.3.

Comment: Even if we did not formally describe how a multistage game is mapped to an equivalent extensive game, it can be seen without difficulties that the subgames defined above correspond to the subgames of an equivalent extensive form.

3.3 Subgame perfectness

We continue to look at a subgame G^u of G. The set of all preplays in U^U at which player i is active is denoted by P_i^u. The restriction of a behavior strategy b_i for G to P_i^u is a behavior strategy b_i^u for the subgame G^u. We say that b_i^u is the strategy *induced* by b_i on G^u. Similarly a strategy combination $b^u = (b_1^u,..., b_n^u)$ is *induced* by $b = (b_1,...,b_n)$ if for i = 1,..., n the behavior strategy b_i^u is induced by b_i on G^u. An i-incomplete strategy combination b^u_{-i} is *induced* by an i-incomplete strategy combination b_{-i} if every component of b^u_{-1} is induced by the corresponding component of b_{-i}. A set E^u of strategy combinations for G^u is *induced* by a set E of strategy combinations for G , if E^u is the set of all strategy combinations b^u induced by strategy combinations $b \in E$.

Let r_i be a best reply to a strategy combination $b = (b_1,...,b_n)$ of G. For every subgame G^u of G let r_i^u, be the strategy induced by r_i on G^u and $b^u = (b_1^u,..., b_n^u)$ the strategy combination induced by b on G^u. We say that r_i is a *subgame perfect best reply* to b if for every subgame G^u of G the behavior strategy r_i^u, is a best reply to b^u.

An equilibrium $\tilde{b} = (\tilde{b}_1,...,\tilde{b}_n)$ is *subgame perfect* if it induces an equilibrium on every subgame G^u of G or, in other words, if for i = 1,...,n the behavior strategy $\tilde{b}_i$ is a subgame perfect best reply to $\tilde{b}$. Similarly an equilibrium set E of *G* is *subgame perfect* if it induces an equilibrium set E^U on every subgame G^u of G.

4. *Delay supergames*

In this section delay supergames will be formally defined. A game of this kind is associated with a bounded multistage game G. In addition to the underlying game G the description of a delay supergame involves further specifications. It is necessary to specify delays after which decisions made on different stages become effective, stopping probabilities as a function of time and initial conditions on what is carried over from the past.

The interpretation of a delay supergame will focus on the case that in the underlying bounded multistage game K decision variables are fixed one after the other by some or all players in K successive stages. The definition of a multistage game does not exclude the possibility that different kinds of decisions have to be made at different preplays of the same length. However, in such cases it may not be adequate to assume that the delay after which a decision becomes effective depends only on the length of the preplay at which it is made.

In section 4 all definitions refer to a fixed but arbitrary bounded multistage game G = (s, A, p, h) in which the maximum length of a play is K. A delay supergame of G is itself also a multistage game G* = (s*, A*, p*, h*) whose constituents are derived in a systematic manner from G and some additional specifications to be explained in 4.1 - 4.4.

4.1 Delay vector

In order to describe a delay supergame of G, it is necessary to specify a *delay vector*

$$m = (m_1, ..., m_k)$$

whose components are non-negative integers with $m_j > m_k$ for $j < k$:

$$m_1 > m_2 > ... > m_K \geq 0$$

The number m_k is called the *delay of stage* k. It is interpreted as the number of periods after which a decision on stage k becomes effective.

4.2 Stopping rule

A delay supergame of G begins with an *initial period* t_o and is played either for at most finitely many periods $t_o, ..., T$ or for potentially infinitely many periods $t_o, t_o + 1, ...$ In the first case the delay supergame is *bounded* and in the second case it is *unbounded.* In the bounded case a *reachable* period is one of the periods $t_o, ..., T$. In the unbounded case all periods $t_o, t_o + 1, ...$ are reachable. The set of all *reachable* periods is denoted by R.

A *stopping rule* w assigns a *stopping probability* w_t with

$$0 \leq w_t < 1$$

to every reachable period $t \in R$. The stopping probability w_t is interpreted as the conditional probability with which the delay supergame stops in period t, if period t-l has been reached. As will be explained later, no payoffs are obtained for period t if the game stops in period t. The players do not know whether the game stops in t when they make their choices at t. This depends on a random choice at t. All choices at t including this random choice are thought of as being made simultaneously. In the bounded case the game always stops after period T, if it is reached.

We think of w as a function together with the set R on which it is defined.

In the bounded case, T will always denote the *last reachable period.* In the unbounded case we shall always assume that w satisfies the following stopping requirement.

Stopping requirement: A real number δ with $0 < \delta < 1$ and a positive integer μ exist, such that the following is true:

$$w_t \geq \delta \quad \text{for all periods} \quad t = t_o + \mu,\ t_o + \mu + 1, \ldots$$

As we shall see later, this stopping requirement secures the random stopping condition of 2.2 for unbounded delay supergames.

4.3 The initial status assignment

In the following m will always be a fixed but arbitrary delay vector, t_o will stand for the initial period, and w will be a fixed but arbitrary stopping rule.

As in 2.7 the set of all preplays u of G with $|u| = k$ is denoted by Q_k and Z_k stands for the set of plays z of G with $|z| = k$. For $k = 0,\ldots K$ a *k-prearrangement* is either a preplay of G of length k or a play z of G with $|z| \leq k$. The set of all k-prearrangements is denoted by S_k. We say that a choice at a preplay u of G is made on *stage* k or that it is a *stage* k *decision* if u belongs to Q_{k-1}.

Consider a delay m_k and a reachable period $t_o+\tau$ with $t < m_k$ The interpretation of m_k as the delay until a stage-k-decision becomes effective suggests that a stage k decision for period $t_o+\tau$ should not be modelled as being made within the delay supergame but rather as predetermined by the past. Accordingly the definition of a delay supergame associated with G requires the specification of all decisions of this kind in a way which will be explained below.

For $\tau = 0,1,\ldots$ we define a *predetermination span* $\delta(\tau)$ interpreted as the highest stage k for which decisions for a period $t_o+\tau$ are excluded by $\tau < m_k$. For $\tau = 0,\ldots, m_1 - 1$ the predetermination span $\delta(\tau)$ is the index k of the smallest delay m_k with $\tau < m_k$. For $\tau = m_1, m_1 + 1,\ldots$ we define $\delta(\tau) = 0$.

An *initial status assignment* assigns an *initial status*

$$x(t,s^*) \in S_\tau \text{ with } \tau = \delta(t-t_o)$$

to every reachable period t. The initial status $x(t,s^*)$ is interpreted as the description of what is predetermined at period t. What is predetermined must be a $\delta(t-t_o)$-prearrangement. Obviously we have:

$$x(t,s^*) = s \quad \text{for } t \geq t_o + m_1.$$

It is possible that x (t,s^*) is a play of length K. This may happen for $t-t_o < m_K$. Later we shall also define $x(t, u^*)$ for every path u^* and for every period $t = t_o, t_o+1,\ldots$ There, too, $x(t, u^*)$ will describe what is predetermined for period t, once u^* has been played in the delay supergame.

The notation $x(\cdot, s^*)$ will be used for the initial status assignment. In the following $x(\cdot, s^*)$ will always be a fixed but arbitrary initial status assignment fitting the delay vector m and the stopping rule w.

4.4 *Initial payoff vector*

As in an ordinary supergame in a delay supergame payoffs for the periods played are accumulated as the game goes on. However, it will be convenient to permit the possibility that some fixed payoffs will be earned in addition to this. One may think of these payoffs as carried over from the past in a similar fashion as the initial status assignment $x(\cdot,s^*)$. Such payoffs carried over from the past arise naturally in subgames of delay supergames.

An *initial payoff vector*

$$c = (c_1,..., c_n)$$

is an n-vector with real valued components. c_i is called *player i's initial payoff:* In the following c will always be a fixed but arbitrary initial payoff vector. The inclusion of an initial payoff vector among the specifications of a delay supergame has the purpose to define a delay supergame in such a way that the concept also covers the subgames of delay supergames.

4.5 *The choice set function of the delay supergame*

In 4.1 - 4.4 we have explained what has to be specified in order to describe a delay supergame associated to a bounded multistage game G: A delay vector m, an initial period t_o, a stopping rule w, an initial status assignment $x(\cdot,s^*)$, and an initial payoff vector c determine a delay supergame $G^*= \Gamma(G, m, w, x(\cdot,s^*), c)$. One may look at Γ as a function which assigns a multistage game G* to every bounded multistage game G augmented by the additional specifications shown as arguments of Γ. This will be made precise below.

The upper index * will be used wherever notation aims at details connected to G* which have a counterpart in G, e.g. preplays, plays, etc. . The star will not be used for symbols like m, w, x, and c which need not be distinguished from corresponding objects in G.

The start s* is a symbol which represents the situation before period t. We now recursively define the choice set function A*, the path set U*, and the status function $x(\cdot,\cdot)$ which assigns a path of G to every pair (t, u*) of a reachable period t and a path $u^* \in U^*$. It is clear that s* is a path and that for all reachable t the status x(t, s*) is already given by the initial status assignment. The recursive definition rests on this basis.

For i = 0,...,n and for every path u* of G* let $D_i(u^*)$ be the set of all reachable periods of the form $t_o+|u^*|+m_k$ with k = 1,..., K for which $A_i(x(t,u^*))$ is non-empty. $D_i(u^*)$ is interpreted as the set of all periods for which i has to make a decision at u*, if u* is a preplay. We refer to the elements of Di(u*) as the *aim periods* and to Di(u*) as the *aim period* set at u*. The union of all

$D_i(u^*)$ with i = 0,...,n is denoted by $D(u^*)$. This set is called the *joint aim period set* at u*.

For i = 0,...n and every path u* let $A^*{\cdot}_i(u^*)$ be the set of all systems of the form

$$a_i^* = (a_i^t)_{t \in D_i(u^*)} \text{ with } a_i^t \in A_i(x(t, u^*)) \text{ for all } t \in D_i(u^*)$$

If u* is a preplay, then the elements a_i^* of $A.{}_i^*$ are *choices* of player i and in the case of a personal player i, all choices of player i. The component $a_i t$ of a_i^* is called player i's *decision* for t *at* u *. The random player has an additional choice ω, the *stopping choice,* at every preplay. If $A{\cdot}_o(u^*)$ is empty the random player also has a *continuation choice* $\tilde{\omega}$

$$A_o^*(u^*) = \omega \cup A{\cdot}_o(u^*) \text{ if } A{\cdot}_o(u^*) \neq \emptyset$$
$$A_o^*(u^*) = \{\omega, \tilde{\omega}\} \text{ if } A{\cdot}_o(u^*) = \emptyset$$

for every preplay u*. The *choice sets* of the personal players are

$$A_i^*(u^*) = A{\cdot}_i(u^*) \text{ for } i = 1,...,n$$

for every preplay u*. For a play z* we have:

$$A_i(z^*) = \emptyset \text{ for } i = 0,...,n$$

It still needs to be explained what distinguishes a play z* from a preplay u*. A play z* must be of the form

$$z^* = u^*a^* \quad \text{with } a^* \in A^*(u^*)$$

where u* is a preplay. A path z* of this form is a *play* if and only if one of the following conditions is satisfied.

(1) a* has the random component $a_o^* = \omega$

(2) a* does not have the random component $a_o^* = \omega$, the game G* is bounded and $t_o + |z^*| = T+1$ is unreachable

In case (2) we speak of a *maximal* play and in case (1) of a *submaximal* play. In unbounded delay supergames plays cannot end in any other way than by a random choice $a_o^* = \omega$. However, in a bounded delay supergame a play can extend over all $T-t_o+1$ reachable periods $t_o,...,T$. The length of a maximal play is $|z^*| = T-t_o+1$.

The periods of a play z* in which the random choice was not ω are called *unstopped*. The last period $t_o + |z^*| -1$ of a submaximal play is called *stopped*. The set of all unstopped periods of a play z* is denoted by L(z*). The set L(z*) can be empty. This happens if already t_o is stopped. In 4.7 it will be explained that in the course of a play payoffs are accumulated for unstopped periods only.

In order to complete the definition of the choice set function we still have to describe how the status of a reachable period t changes in the course of playing the game. For every path of the form u*a* the *status* x(t,u*a*) of t at u*a* is recursively defined as follows:

$$x(t,u^*a^*) = x(t,u^*)a^t \text{ with } a^t \text{ in } a^* \text{ for } t \in D(u^*)$$

$$x(t,u^*a^*) = x(t,u^*) \text{ for } t \notin D(u^*)$$

for every preplay u* and every a* ∈ A* (u*). This means that the status of t at u* is changed by the decisions for t in a* but, of course, only if t is an aim period of u*. It is clear how the choice set function A*, the preplay set U*, the preplay set Z*, and the *status function* x(·,·) are determined by the joint recursive definition given above.

4.6 The probability assignment of the delay supergame

The probability of a random choice a_o* at a preplay u* according to the probability assignment p* of G* may be thought of as the result of independent random draws according to p for all aim periods in D_o(u*) combined with an independent decision on whether to stop with probability $w_{|u^*|}$ or to continue. In order to make this more precise consider a random choice a,* at a preplay u* of G*. For every t ∈ D_o(u*) let π^t be the probability

$$\pi^t = p_u(a_o^t) \quad \text{with } u = x(t,u^*) \text{ and } a_o^t \text{ in } a_o^*$$

Moreover let π be the product of all π^t with t ∈ D_o(u*). Then we have:

$$p_u^*(\omega) = w_{|u^*|}$$
$$p_u^*(\varpi) = 1-w_{|u^*|} \text{ for } A\cdot_o(u^*) = \emptyset$$
$$p_u^*(a_o^*) = (1-w_{|u^*|})\pi \text{ for } a_o^* \in A\cdot_o(u^*)$$

It can be seen without difficulty that the random stopping requirement imposed on w in 4.2 secures the random stopping condition of 2.2 for p*.

4.7 The payoff function of the delay supergame

A path u* of G* has the form of a sequence starting with s* and continuing with |u*| choice combinations at the periods $t_o,\dots,t_o+|u^*|-1$. Accordingly we say that periods t with $t_o \le t < t_o+|u^*|$ are *in the past* of u*. The other reachable periods from $t_o+|u^*|$ on are *in the future* of u*. All decisions for a period t in the past of u* which may have to be made as long as the status of t is a preplay, must be made before t or at t. Therefore the status x(t,u*) of a period in the past of u* must be a play of G.

In the delay supergame G* payoffs for a play z* are composed of initial payoffs and of payoffs for unstopped periods accumulated as z* is played:

$$h^*(z^*) = c + \sum_{t \in L(z^*)} h(x(t,z^*))$$

No payoffs are obtained for a *stopped* period, in which the random choice was ω. The choice of ω is thought of as immediately effective in the sense that the period is stopped already at its beginning.

From what has been explained in 4.5 and 4.6 it is clear that s*, A*, and p* satisfy the conditions jointly imposed on the start, the choice set function and the probability assignment of a multistage game. In order to see that G*= (s*,A*,p*,h*) has all the properties of a multistage game it remains to show that h* satisfies the boundedness condition of 2.3.

Since G is bounded and K is the maximum length of a play z of G it follows by the boundedness condition for G that we have

$$|h_i(z)| \leq C_o + KC_1$$

for some constants C_o and C_1. Let C_1^* be the right hand side of this inequality and let C_o^* be the maximum of the $|c_i|$. Obviously h* satisfies the boundedness condition with C_o^* and C_1^* in the place of C_o and C_1.

We have now completed the definition of the delay supergame

$$G^* = (s^*,A^*,p^*,h^*) = \Gamma(G,m,w,x(\cdot,s^*),c)$$

and we have shown that G* is a multistage game.

4.8 Expected payoffs in delay supergames

The expected payoff vector H*(b*) for a strategy combination b* = $(b_1^*,...,b_n^*)$ of G* is defined in the same way as for multistage games in general. However, it will be useful to express H*(b*) in a way which focuses on decisions for periods rather than on choices at periods.

For every reachable period t let V_t^* be the set of all paths v* with |v*| = $t-t_o+1$. Obviously a path $v^* \in V_t^*$ ends with a choice combination at period t. After period t all decisions for t have been made. Therefore the status x(t,v*) for $v^* \in V_t^*$ must be a play $z \in Z$. For every $z \in Z$ let Vt*(z) be the set of all $v^* \in V_t^*$ with x(t,v*) = z. For every strategy combination b* = $(b_1^*,...,b_n^*)$ and every reachable period $t \in R$ define

$$b^*(t,z) = \sum_{v^* \in V_t^*(z)} b^*(v^*)$$

and

$$F^t(b^*) = \sum_{z \in Z} b^*(t,z)h(z)$$

We call $b^*(t,z)$ the *realization probability of z in period* t *under* b^* and $F^t(b^*)$ the *period payoff vector of t* for b^*. The component $F_i^t(b^*)$ is player i's *period payoff of* t. Obviously we have:

$$H^*(b^*) = c + \sum_{t \in R} F^t(b^*)$$

for every strategy combination b^* in G^*

4.9 The subgames of a delay supergame

Let s^+ be a preplay of G^* with $|s^+| \geq 1$ and let

$$G^+ = (v^+, A^+, p^+, h^+)$$

be the subgame of G^* at s^+. It can be seen without difficulty that G^+ is a delay supergame of G: The game G^+ starts with period $t^+ = t_o + |s^+|$. The delay vector m is the same one as in G^*. The stopping rule w^+ is the restriction of w to the reachable periods $t_o^+, t_o^+ + 1, \ldots$. The initial status assignment of G^+ is $x(\cdot, s^+)$. The initial payoff vector C^+ of G^+ is as follows:

$$c^+ = c + \sum_{t = t_o}^{t_o^+ - 1} h(x(t,s^+))$$

We can write $G^+ = \Gamma(s^+, m, w^+, x(\cdot,s^+), c^+)$.

5. Relationship between subgame perfect equilibria in bounded multistage games and their delay supergames

On the basis of the definitions given in previous sections, it will now be possible to make two statements about the relationship between subgame perfect equilibria in bounded multistage games and their associated delay supergames. These statements provide a precise interpretation of the non-cooperative analysis of a bounded multistage game model in terms of its associated delay supergames.

As before G = (s, A, p, h) will be a bounded multistage game and $G^* = (s^*, A^*, p^*, A^*)$ will be one of its delay supergames with the additional specifications m, w, $x(\cdot, s^*)$, and c.

5.1 Generated strategies

Let b_i be a behavior strategy of player i for G. The behavior strategy b_i^* *generated* by b_i in G^* is defined as follows

$$b_i^* = (b_{iu^*})_{u^* \in P_i^*}$$

with

$$b_{iu^*}(a_i^*) = \prod_{t \in D_i(u^*)} b_i(a_i^t) \text{ with } a_i^t \text{ in } a_i^*$$

for every $a_i^* \in A_i(u^*)$. This means that at u* decisions for the aim periods are chosen by independent draws with the appropriate probabilities specified by b_i. It is clear that b_i^* has the properties of a behavior strategy for G*.

We say that $b^* = (b_1^*,..., b_n^*)$ is *generated* by $b = (b_1,..., b_n)$ if for i = 1,..., n the behavior strategy b_i^* is generated by b_i. Similarly an i-incomplete strategy combination b_{-i}^* is *generated* by b_{-i}, if the components of b_{-i}^* are generated by the corresponding components of b_{-i}.

The set of all preplays v* of G^+ with $|v^*| = t-t_o$ is denoted by Q_t^*. (This definition is not exactly analogous to that of Q_t in *2.7.)* The preplays in Q_t^* are those which are followed by choices at period t. We shall have to look at the probability that one of these preplays is reached and that then the game is not stopped in period t. This probability, denoted by W_t is the product of the $t-t_o+1$ terms l-w, with $\tau = t_o,...,t$. Obviously a payoff for period t is obtained with probability W_t. Therefore we call W_t the *payoff probability* of t.

For every reachable period $t \in R$ the subgame of G at the initial status x (t, s*) will be denoted by $G^t = (s^t, A^t, p^t, h^t)$. Of course, s^t is just another name for x (t, s*). Let $b = (b_1,...,b_n)$ be a strategy combination for G and for every $t \in R$ let $b^t = (b_1^t,...,b_n^t)$ be the strategy combination induced by b on G^t. Moreover let b* be the strategy combination generated by b in G*. Then we have:

$$b^*(t, z) = W_t b^t(z)$$

for every play z of G^t. Therefore for every $t \in R$ the period payoff vector of t for b* is as follows:

$$F^t(b^*) = W_t H^t(b^t)$$

where H^t is the expected payoff function for G^t. This yields

$$H^*(b^*) = \sum_{t \in R} W_t H^t(b^t)$$

In this way expected payoffs for strategy combinations generated in delay supergames can be expressed as payoffs in subgames of the underlying bounded multistage game.

5.2 Generated subgame perfect best replies

In this section we prove a lemma which leads to the conclusion that a subgame perfect equilibrium generates a subgame perfect equilibrium.

Lemma 1: Let $b = (b_1,\ldots, b_n)$ be a strategy combination for G and let $b^* = (b_1^*,\ldots,b_n^*)$ be the strategy combination generated by b in G*. Moreover let r_i be a subgame perfect best reply to b and let r_i^* be generated by r_i in G*. Then r_i^* is a subgame perfect best reply to b*.

Proof Let b_{-i} and b_{-i}^* be the i-incomplete strategy combinations whose components are in b and b* respectively. Let f_i^* be any behavior strategy for G*. We are interested in player i's period payoffs $F_i^t(f_i^* b_i^*)$. As before, Q_t^* is the set of all preplays v* with $|v^*| = t\text{-}t_o$. For every period $t \in R$, every preplay $v^* \in Q_t^*$, every preplay u of G^t and every choice $a_i \in A_i(u)$ let

$$f_{iu}^{v^*}(a_i)$$

be the probability with which a_i is chosen as player i's decision for t at the appropriate place on the path v*. Obviously f_{iU}^{v*} is a local strategy at u. Let $f_i^{v^*}$ be the behavior strategy for G^t which assigns these local strategies to the preplays of G^t. For every preplay $v^* \in Q^*$ let $\varphi(v^*)$ be the conditional probability that v* is realized by $f_i^{v^*}b_{-i}^*$ under the condition that a preplay in Q* is reached. This conditional probability is well defined, since every reachable period is reached with positive probability. We have:

$$F_i^t(f_i^* b_{-i}^*) = W_t \sum_{v^* \in Q_t^*} \varphi(v^*)\, H_i^t(f_i^{v^*} b_{-i})$$

Since r_i is a subgame perfect best reply to b_{-i} we have:

$$H_i^t(f_i^{v^*} b_{-i}) \leq H_i^t(r_i\, b_{-i})$$

In view of the fact that $\varphi(\cdot)$ is a probability distribution over Q_t^* this yields

$$F_i^t(f_i^* b_{-i}^*) \leq W_t\, H_i^t(r_i\, b_{-i}) = F^t(r_i^* b_{-i})$$

This is true for every reachable period t. Therefore we can conclude that

$$H^*(f_i^* b_{-i}^*) \leq H^*(r_i^* b_{-i}^*)$$

holds for every strategy f_i^* for G*. Consequently r_i^* is a best reply to b*. Since the subgames of G* are delay supergames, the same argument can be applied to all subgames of G*. This shows that r* is a subgame perfect best reply to b*.

5.3 Subgame perfect equilibria generate subgame perfect equilibria

The first main conclusion of this paper is the following theorem.

Theorem 1: Let b be a subgame perfect equilibrium of a bounded multistage game G and let b* be the strategy combination generated by b in a delay supergame G^* of G. Then b^* is a subgame perfect equilibrium of G^*.

Proof: The assertion is an immediate consequence of the lemma of 5.2

5.4 Determinate multistage games

A multistage game is called *determinate,* if it has at least one subgame perfect equilibrium and if, in addition to this, the set of all its subgame perfect equilibria is a subgame perfect equilibrium set. In the following we shall introduce some definitions and notations connected to determinate bounded multistage games.

Let G be a determinate multistage game. Let E be the set of all subgame perfect equilibria of G. Since G is determinate, E is a subgame perfect equilibrium set of G. For every subgame G^u of G at a preplay u of G let E^u be the set induced by E on G^u. Since E is a subgame perfect equilibrium set, E^u is a subgame perfect equilibrium set, too. Moreover E^u is the set of all subgame perfect equilibria of G^u. For every preplay u of G let

$$e(u) = (e_1(u),...,e_u(u))$$

be the common expected payoff vector $H^u(b^u)$ for all $b^u \in E$,. The game G itself is a subgame of G at the start s of G. Accordingly $e(s)$ is the common payoff vector $H(b)$ for all $b \in E$. For the case that u is a play define $e(u) = h(u)$. We call *e(u)* the *replacement payoff vector* at u and refer to $e_i(u)$ as the *replacement payoff of* player i.

The name "replacement payoff" suggests itself by the use of which we are going to make of $e(u)$. For every preplay v of G we construct a *truncated sub game* G,. This game G_v results from the subgame G^v as follows: Every subgame G^u of G^v with $u = va$ and $a \in A(v)$ is replaced by the payoff vector $e(u)$. This means that all preplays $u = va$ are plays of G_v with the payoff vector

$$h_v(a) = (h_{v1}(a),..., h_{vn}(a)) = e(va)$$

It can be seen without difficulty that a subgame perfect equilibrium b^v of G^v induces an equilibrium on G_v. We shall make use of a simple fact expressed by the following lemma.

Lemma 2: Let $r_v = (r_{v1},..., r_{vn})$ be an equilibrium of a truncated subgame G_v of a determinate multistage game G. Then we have

$$H_v(r_v) = e(v)$$

where Hv is the expected payoff function of G_v and $e(v)$ is the replacement payoff vector at v in G.

Proof: Let $b^v = (b_1{}^v,..., b_n{}^v)$ be a subgame perfect equilibrium of the subgame G^v. Let r^v be the strategy combination whose components $r_i{}^v$ assign the local strategy r_{iv} to v and agree with $b_i{}^v$ everywhere else. It can be seen immediately that r^v is a subgame perfect equilibrium of G^v. It is also clear that the payoff vector for r^v in G^v is nothing else than the payoff vector for r_v in G_v. In view of the determinateness of G we have

$$H_v(r_v) = H^v(r^v) = e(v)$$

for every preplay v of G.

5.5 Determinateness of bounded delay supergames of determinate bounded multistage games

The following theorem is the second main conclusion of this paper.

Theorem 2: Every bounded delay supergame of a determinate bounded multistage game is determinate.

Proof: Let G be a determinate bounded multistage game and let G* be one of its bounded delay supergames. Since G is determinate, G has at least one subgame perfect equilibrium. Let r be a subgame perfect equilibrium of G and let r* be the strategy combination generated by r in G*. In view of theorem 1, this strategy combination r* is a subgame perfect equilibrium of G*. It remains to show that all subgame perfect equilibria of G* form a subgame perfect equilibrium set. We shall prove this by induction on the number T-t,+ 1 of the reachable periods t_o... T of G*.

Consider first the case $T-t_o+1 = 1$ in which $t_o = T$ is the only reachable period. The game ends after the choice combination at t_o. All plays have the form s*a with $a \in A(s^*)$. A strategy combination for G* is a strategy combination for G_v with $v = x(t_o, s^*)$. It follows by lemma 2 that the same expected payoff vector is obtained for all equilibria of G*. Consequently G* is determinate.

From now on assume that the number of reachable periods in G* is greater than 1 and that the assertion holds for all delay supergames with a smaller number of reachable periods.

As above let r be a subgame perfect equilibrium of G and let r* be generated by r in G*. All subgames of G* at preplays of the form s*a* with $a^* \in A^*(s^*)$ are determinate since they have a smaller number of reachable periods. A strategy b_i^* of a personal player i for G* is called *semigenerated* by r if for all preplays v* of G* with $|v^*| > 0$ the local strategy assigned to v* is the local strategy assigned by r*. In view of the determinateness of the subgames of G* starting with period t_o+1 the payoffs in these subgames and their subgames do not depend on the particular subgame perfect equilibrium played. Therefore it is sufficient to show that the set of all equilibria in strategies semigenerated by r is an equilibrium ser.

It follows by lemma 1 that it is always possible to find a semigenerated subgame perfect best reply to a combination of semigenerated strategies for G*.

It can be seen easily that in the case of a combination of strategies semigenerated by r, a subgame perfect best reply can be found among the strategies semigenerated by r.

In order to show that the set of all equilibria in strategies semigenerated by r is an equilibrium set it is sufficient to look at the following *truncated game* G+ of G*. This game G+ results from G* by the replacement of all subgames at preplays of the form s*a* with $a^* \in A^*(s^*)$ by their subgame perfect equilibrium payoff vectors. This game G^+ is a multistage game with only one stage. The pure strategies of the personal players in G^+ are their choice sets $A_i^*(s^*)$. For the sake of shortness we write A_i^+ instead of $A_i^*(s^*)$ for i = 0,..., n and A+ instead of A* (s*) .

It can be seen immediately that the subgame perfect equilibria in strategies semigenerated by r form an equilibrium set if the set of all equilibria of the truncated game G^+ is an equilibrium set. It remains to show that this is the case. For this purpose we shall look at the expected payoff $H^+(a^*)$ for a choice combination $a^* \in A+$. Let N^+ be the set of players active at s* in G*. For i = O,..., n let D_i^+ be the aim period set $D_i(s^*)$ and let D+ be the aim period set D(s*).

Let $a^* \in A^*$ be a choice combination. For every $i \in N^+$ let a_i^* be the choice of i in a* and for every $t \in D_i^+$ let ait be the decision for t specified by a_i^* and at the decision combination specified by a* for t. After the choice of a* every reachable period has the status x (t,s*a*) Later decisions for t are made with the probabilities required by r. Therefore the expected period payoff vector of t for a* is e(x(t,s*a*)) multiplied by W_t. This yields

$$H^+(a^*) = c + \sum_{t=t_0}^{T} W_t \, e(x(t,s^*a^*))$$

We have

$$x(t,s^*a^*) = x(t,s^*)a^t \text{ for } t \in D^+$$
$$x(t,s^*a^*) = (t,s^*) \text{ for } t \in R \setminus D^+$$

The period payoff vectors for $t \in R \setminus D+$ do not depend on a*. Let J be the sum of all these period payoff vectors and c. We obtain

$$H^+(a^*) = J + \sum_{t \in D^+} W_t e(x(t,s^*)a^t)$$

Let $f = (f_1,..., f_n)$ be a strategy combination for G+. For every $i \in N^+$, every $t \in D^+$, and every $a_i^t \in A_i(x(t,s^*))$ let $f_i^t(a_i^t)$ be the probability that a choice a_i^* is taken by f_i which specifies $a_i t$ as the decision for t. The function f_i^t defined in this way is a behavior strategy for the truncated subgame of G at x(t,s*). The symbol G_t is used for this game. Let $f^t = (f_1^t,...,f_n^t)$ be the strategy combination containing these behavior strategies and the empty strategies of the players not active at x(t,s*). The conditional realization probability $f^t(a^t)$ of a^t at x(t,s*) is the probability that a choice combination a* results from the use of the strategies in f, such that a* specifies the decision combination a^t for period t. We can now express the expected payoff vector for f in G^+ as follows:

$$H^+(f) = J + \sum_{t \in D^+} W_t \sum_{a^t \in A(x(t,s^*))} f^t(a^t)e(x(t,s^*)a^t)$$

Let f = (f_1,..., f_n) be a strategy combination for G^+. It will now be shown that the following is true: if f is an equilibrium of G^+, then for every t ∈ D+ the strategy combination f^t must be an equilibrium of G^t. In order to see this, suppose that for one t ∈ D+ this is not the case. Then in G_t one personal player i has a pure best reply $\tilde{a}_i^t$ which yields a better payoff against the strategies of the others in f^t than f_i^t does. If this is the case, player i can improve its payoff by the following change of behavior: If f_i selects a choice a_i* which does not specify $\tilde{a}_i$ for t, then the choice specified by a_i* for t is replaced by $\tilde{a}_i^t$ and the changed choice resulting from a_i* in this way is taken. Otherwise the behavior prescribed by f_i remains unchanged. The behavior strategy g_i thereby obtained improves player i's period payoff for t and lets other period payoffs of player i unchanged.

It is a consequence of the determinateness of G that for every t ∈ D+ the set of all equilibria of G_t is an equilibrium set. Otherwise it would be possible to construct two subgame perfect equilibria of the subgame G^t at x(t,s*) of G with different payoff vectors. We can conclude that the set of all equilibria of G^+ is an equilibrium set. This completes the proof of the theorem.

6. Discussion

After the concepts and conclusions of this paper have been made precise their significance for economic theory will be discussed in this section.

6.1 The relationship between multistage games and their delay supergames

Multistage game models often do not really aim at one shot strategic interactions in which stage decisions are made once and for all in a strict temporal order, but rather ongoing situations, in which decisions represented as made at earlier stages are thought of as more long term than those made on later stages. What is the essential difference between more long term and more short term decisions? It has been argued in the introduction that in many cases the distinguishing feature seems to be the length of the delay until a decision becomes effective. The formal elaboration of this idea leads to the notion of a delay supergame of a multistage game.

Of course, other differences between more long term and more short term decisions may also enter the picture in specific applications. It is not clear to what extent additional distinguishing features like greater change costs for more long run decisions would force us to modify our conclusions. This is a question which needs to be explored with the help of more general dynamic game models associated to multistage games. No attempt in this direction can be made here. The conclusions drawn in this paper cannot claim to achieve more than a precise interpretation of subgame perfect equilibria in multistage game models which do not permit a direct temporal interpretation of the order of stages. In this way the gap between model and reality is not closed, but maybe diminished or at least illuminated.

6.2 Consequences of the assumption of full rationality

In the remainder of the paper we shall always look at a bounded multistage game and one of its delay supergames. We first discuss the consequences of the assumption that the delay supergame is played by fully rational players with common knowledge about its rules. In the case of a unique subgame perfect equilibrium set of the multistage game and a bounded delay supergame, theorem 2 yields the conclusion that the behavior to be expected in the delay supergame is essentially correctly described by the perfect equilibrium set of the multistage game. In this case not all subgame perfect equilibria of the delay supergame are generated by those of the multistage game, but this does not matter as far as payoffs in the whole game and all its subgames are concerned.

In all other cases we cannot say more than what has been shown by theorem 1. Every subgame perfect equilibrium of the multistage game generates a subgame perfect equilibrium of the delay supergame. In addition to this the delay supergame may have many other equilibria some of which may yield higher payoffs for all players. In such cases the non-cooperative analysis of the multistage game fails to reveal the potential for quasicooperative subgame perfect equilibrium behavior in the delay supergame. This raises a problem with respect to the interpretation of subgame perfect equilibria of the multistage game.

6.3 The finiteness argument

An upper bound, say a million years, can be named for the survival of any economic situation. Therefore unbounded delay supergames should be looked upon as simplified descriptions of long lasting ongoing situations of finite duration. Conclusions which rest on the feature of unboundedness cannot be taken seriously. In this way the non-cooperative analysis of a multistage game with a unique subgame perfect equilibrium set can be defended as a satisfactory substitute for the full treatment of the delay supergame.

However, is this argument really valid? Experimental results clearly show, that experienced players cooperate in 10-period supergames of the prisoner's dilemma game until shortly before the end (Selten and Stoecker 1986) in spite of the fact that this is excluded by subgame perfect equilibrium.

6.4 Slightly incomplete information

In order to deal with the phenomenon of cooperation in finite prisoner's dilemma supergames without an essential relaxation of rationality assumptions one can take the point of view that the supergame is not the game really played, and that instead of this the players are involved in an similar, but different game. Kreps, Wilson, Milgrom and Roberts (1982) assume that with a small probability a player may be of a different type, with preferences which make it profitable to take the cooperative choice as long as the opponent has been observed to do this. The introduction of such types transforms the supergame to a game with slightly incomplete information, and thereby opens quasicooperative opportunities similar to those in the case of infinite repetition.

The approach of Kreps, Wilson, Milgrom and Roberts (1982) is not a convincing explanation of the experimental evidence. If cooperation in the finitely repeated prisoner's dilemma were the result of sophisticated rational deliberation, then it should be observed already in the first supergame played and not only after a considerable amount of experience as it happens in our experiments (Selten and Stoecker 1986). Kreps and Wilson (1982) have applied the same incomplete information approach to the chain store paradox (Selten 1978). Here, too, the experimental evidence points in a different direction (Jung, Kagel, and Levin 1994). The bounded rationality approach of my (1978) paper seems to be in better agreement with the data.

6.5 Institutional reasons for the absence of cooperation

From what has been said up to now, it is clear that, even if the assumptions of theorem 2 are satisfied, we cannot rely on the descriptive validity of the non-cooperative analysis of the multistage game as a substitute for the full treatment of the delay supergame. However there may be institutional reasons for the exclusion of cooperation or subgame perfect quasicoope-ration in the supergame.

Consider the case of an oligopoly in an economy with strictly enforced cartel laws and assume that the oligopolistic market is adequately modelled as a delay supergame of a bounded multistage game. Presumably the cartel laws forbid collusion, but what does this mean? Obviously the cartel laws cannot simply demand that a subgame perfect equilibrium of the delay supergame is played. This may fail to exclude subgame perfect quasicooperation.

It seems to be natural to define the absence of collusion as a state of affairs, in which a subgame perfect equilibrium of the delay supergame is played which is generated by a subgame perfect equilibrium of the underlying multistage game. If the absence of collusion in this sense is effectively enforced, then the non-cooperative analysis of the multistage game is a satisfactory substitute for that of the delay supergame.

However, against this argument the objection can be raised, that it remains unclear, how the cartel laws are enforced. Presumably the enforcement agency does not have the same knowledge of the market which is available to the oligopolists. The same may be true for the court to whom the oligopolists can appeal in the case of a disagreement with the cartel authority. It would be desirable to model the cartel authority explicitly as a player in order to throw light on its strategic interaction with the oligopolists.

In an open oligopoly collusion may have the disadvantage that new entrants are attracted by high profits. This may have the effect that after the entry of new competitors the profits of the incumbents achieved by collusion are lower then they would have been without entry in the absence of collusion. In a paper with the title "Are cartel laws bad for business" (Selten 1984) I have presented a multistage game model which yields the conclusion that under plausible conditions about the distribution of the market parameters total profits of all firms in the economy are increased by effectively enforced cartel laws compared with a situation in which cartels can be legally formed. This effect is due to excessice entry in the absence of cartel laws. Thus the task of the cartel authority may be facilitated by an economy wide advantage for industrial enterprises, even if the firms on some markets would gain by the possibility of legal cartel formation.

6.6 An adaptive interpretation

Experimentally observed behavior is only boundedly rational and the same must be expected of the behavior of firms in real markets. Research on the behavioral theory of the firm (Cyert and March, 1963; Earl, 1988) provides ample evidence for this. In the following I shall try to present some tentative ideas on the consequences of bounded rationality for the interpretation of subgame perfect equilibria of multistage game models and those generated in their delay supergames. Admittedly my remarks in the remainder of the paper will be sketchy and speculative. No attempt is made to support them by a thorough examination of the relevant experimental literature.

Experimental subjects do not compute the solutions of complex optimization tasks in order to make their decisions. Apart from very simple cases it cannot be expected that subgame perfect equilibria are found by rational deliberation. However this does not mean that such equilibria are irrelevant for the prediction of behavior. In repetitive experimental settings equilibria can be learned by adaptation to past experience.

The experimental literature on double auction markets initiated by Vernon Smith (1962, 1964) provides a well known example. Usually competitive equilibrium is reached after a relatively small number of identical repetitions of the same market. The traders do not compute the equilibrium price. Convergence to competitive equilibrium is the result of an adaptive process which adjusts bids and asks to observed imbalances of supply and demand.

The double auction is a very complex game with incomplete information. Behavior does not converge to an equilibrium of this game, but rather to an equilibrium of a fictitious associated game with complete information, in which reservation prices and resale values of all traders are publicly known. Something similar can be observed in Cournot oligopoly experiments in which the producers are not informed about the costs of their competitors (e.g. Sauermann and Selten 1959). It is my impression that convergence to an equilibrium of the fictitious complete information game, which would be played if all private information were public, is favored by the incomplete knowledge about the other players' payoffs. This lack of knowledge hides the quasicooperative opportunities of the complete information game.

In a bounded multistage game played just once, with stages following each other in temporal succession, adaptive adjustment processes cannot work. This is different in sufficiently long delay supergames. Here players repeatedly face the necessity to adapt their decisions on all stages. There is a good chance that, under favorable conditions, subgame perfect equilibria generated by those of the underlying bounded multistage game, are learnt by experience. Conditions are favorable, if the game which is played is not really the delay supergame but a modified version with reduced payoff information in the following sense: The players have little knowledge about the other players' payoff functions and do not observe their payoffs. However as in the unmodified delay supergame they do observe the other players' past choices and they know their own payoff functions.

My thoughts about the descriptive relevance of the non-cooperative analysis of multistage game models can be summarized by the following hypothesis: *Learning in a modified delay supergame with reduced payoff information converges to behavior in agreement with the predictions derived from a subgame perfect equilibrium of the underlying multistage game, the more so, the less the players know about other players' payoffs.*

It is necessary to run experiments specifically designed to test this hypothesis. If the hypothesis succeeds to survive the tests, it will provide descriptive content to the non-cooperative analysis of multistage game models.

6.7 Non-monetary motivation

The hypothesis stated above may have to be weakened in several directions. One difficulty which may arise is the influence of non-monetary motivation. The monetary payoffs offered in an experimental game are often not the only motivational factors present. The ultimatum game has an obvious subgame perfect equilibrium, if only monetary payoffs are considered, but non-monetary motivational forces make it unacceptable for the recipient of the ultimatum (Güth, Schmittberger and Schwarze, 1982; Güth and Tietz 1990). In the case of the ultimatum game the motivational force is resistance to unfairness.

Another motivational factor is clearly visible in an experimental labor market (Fehr, Kirchberger and Riedl 1993). A reciprocity norm induces employees to supply more effort than they have to as a voluntary reward for high wages, in a situation which excludes reputation effects by the experimental set up. This results in considerable deviations from competitive equilibrium.

Much more could be said about the experimental evidence for non-monetary motivations, but this will not be done here. Interestingly resistance to unfairness and reciprocity both depend on at least some knowledge of the other player's payoff. A lack of such knowledge does not only hide quasi-cooperative opportunities but also weakens the force of social norms which facilitate non-equilibrum cooperation. However, one cannot expect that it will always be possible to ignore the influence of non-monetary motivational forces on behavior in modified delay supergames with reduced payoff information.

REFERENCES

Aumann, R.J., 1959, Acceptable Points in General Cooperative n-Person Games, in: R.D. Luce and A.W. Tucker (eds.) *Contributions to the Theory of Games IV* , Ann. Math. Study 40, Princeton, NJ., 287 - 324

Benoit, J.P., and V. Krishna 1985, Finitely Repeated Games, *Econometrica 53, 905 - 922*

Cournot, A., 1838, *Recherches* de la *théorie des richesses,* Paris

Fehr, E., G. Kirchsteiger, and A. Riedl, 1993, Does Fairness Prevent Market Clearing?. An Experimental Investigation, *Quarterly Journal of Economics,* 108, 2, 437 - 460

Fouraker, L.E., and S. Siegel 1963, *Bargaining Behavior; New* York

Friedman, J., 1977, Oligopoly and the *Theory of Games* , North-Holland Publishing Company, Amsterdam-New York-Oxford

Güth W., R. Schmittberger, and B. Schwarze, 1982, An Experimental Analysis of Ultimatum Bargaining, Journal *of Economic Behavior and Organization, 3, 3, 367 - 88*

Güth, W., and R. Tietz, 1990, Ultimatum Bargaining Behavior -A Survey and Comparison of Experimental Results, *Journal of Economic Psychology* 11, 417 - 449

Hoggatt, A.C., 1959, An Experimental Business Game, *Behavioral Science, 4,* 192 - 203

Jung, Y.J., J.H. Kagel, and D. Levin, 1994, On the Existence of Predatory Pricing: An Experimental Study of Reputation and Entry Deterrence in the Chain Store Game, *Rand Journal of Economics 25,* 1, 72 - 93

Kreps, D.M., and J.A. Scheinkman, 1983, Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes, *Bell Journal of Economics,* 14, 326 - 337

Marschak, Th., and R. Selten, 1978, Restabilizing Responses, Inertia Supergames and Oligopolistic Equilibria, *Quarterly Journal of Economics, 92,* 71 - 93

Nash, J.F., 1951, Non-Cooperative Games, *Annals of Mathematics 54, 286 - 295*

Rubinstein, A., 1978, Equilibrium in Supergames with the Overtaking Criterion, *Journal of Economic Theory* 21, 1 - 9

Rubinstein, A., 1980, Strong Perfect Equilibrium in Supergames, *International Journal of* Game *Theory 9,* 1 - 12

Sauermann, H., and R. Selten, 1959, Ein Oligopolexperiment, *Zeitschrift für die gesamte Staatswissenschaft,* 115, 427 - 471, 9 - 59

Selten, R., 1965, Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetragheit, *Zeitschrift für die gesamte Staatrwissenschaft,* 121, 301 - 24, 667 - 89

Selten, R., 1973, A Simple Model of Imperfect Competition where 4 are Few and 6 are Many, *International Journal of* Game *Theory* 2, 141 - 201. (Reprinted in Selten, R., 1988, *Models of Strategic Rationality,* Dordrecht: Kluwer)

Selten, R., 1978, The Chain Store Paradox, *Theory and Decision,* 9, 127 - 159 (Reprinted in Selten, R., 1988, *Models of Strategic Rationality,* Dordrecht: Kluwer)

Selten, R., 1984, Are Cartel Laws Bad for Business? in H. Hauptmann, W. Krelle and K.C. Mosler (eds.) *Operations Research and Economic Theory,* Berlin: Springer Verlag (Reprinted in Selten, R., 1988, Models *of Strategic Rationality,* Dordrecht: Kluwer)

Selten, R., and R. Stoecker, 1986, End Behavior in Finite Prisoner's Dilemma Supergames, Journal of Economic Behavior and Organization, I, 47 - 70

Smith, V., 1962, An Experimental Study of Competitive Market Behavior, *Journal of Political Economy, 70,* 111 - 137

Smith, V., 1964, Effect of Market Organization on Competitve Equilibrium, Quarterly *Journal of Economics 78,* 181 - 201

Stern, D.H., 1967, Some Notes on Oligopoly Theory and Experiments, in M. Shubik (ed.) *Essays in Mathematical Economics,* Princeton, NJ., 255 - 281

von Neumann, J., and O. Morgenstern, 1944, *Theory of Games* and *Economic Behavior;* Princeton University Press, Princeton, NJ.