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Optics of light sources moving in refractive media

Nobel Lecture, December 11, 1958

Peculiarities of radiation in a medium

For a number of years the Vavilov-Cerenkov effect appeared as but a peculiar optical phenomenon difficult to observe. Light emission was induced by using radioactive preparations and the glow was observed visually[1]. The weakness of the glow seemed to preclude any application of the phenomenon in physics, and so much the more in engineering.

Since the theory of the Vavilov-Cerenkov effect appeared[2], the phenomenon could be regarded as an instance of super-light velocity optics*. This was a singular example in this field, which seemingly was isolated from any other known physical phenomenon. It was evident that in principle other manifestations of super-light velocity optics were also possible, but their observation appeared very complicated. For example, the first calculations already indicated that if the Vavilov-Čerenkov radiation were induced not by an electric charge, but, say, by the magnetic moment of an electron, it should be so weak that its experimental detection would not be feasible[3]. It was likewise evident that it would be difficult to create conditions for observation of atoms moving at super-light velocities[4].

Theoretical analysis of all these problems was for a number of years of interest chiefly from the viewpoint of principle.

Progress in nuclear physics and the improvement of experimental techniques in recent years has resulted in the fact that the Vavilov-Čerenkov effect has found numerous applications in the physics of high-energy particles. A connection between this phenomenon and many other problems has also been found, as, for example, the physics of plasma, astrophysics, the problem of radio wave generation, the problem of acceleration of particles, etc.

A broader approach to the treatment of the phenomena related to the

* A summary of the results of theoretical work and list of references are given in the review by B. M. Bolotovsky[8].

Vavilov-Čerenkov effect has now become not only justified but essentially necessary.

The question naturally arises as to the peculiarities of a radiation which may be set up not only by an electric charge, but by any source of light, moving in a refractive mediums. Such a general approach to the problem, involving, notably, the Vavilov-Čerenkov effect, is of interest now not only from the viewpoint of principle. It may be hoped that some phenomena of this range will also become in the near future a subject of experimental study.

Since the discovery of the Vavilov-Čerenkov effect, our ideas of the mechanism of interaction between a rapidly moving particle and a medium have undergone a considerable change.

Formerly it appeared unquestionable that radiation arising during an electromagnetic interaction between high-energy particles and a medium is always some kind of a « bremsstrahlung ». Most of the energy of such radiation is carried by high-energy photons. The optical properties of the medium should not be of significance for the emission and propagation of such photons. It was also assumed that the processes of ionization and excitation by fast particles might be regarded as a sum of independent interactions of such particles with individual atoms and molecules. This led to the deduction that generally for interaction between high-energy particles and a substance its macroscopic properties are likewise of no importance.

The discovery and interpretation of the Vavilov-Čerenkov effect, and then the connection between this phenomenon and ionization losses, found by Fermi[6], have led to a revision of this viewpoint. It has now become evident that the macroscopic properties of the medium play an important part in the processes of radiation of light by rapidly moving particles.

The ratio between the velocity of the emitter and that of light is a highly important factor on which radiation depends. In a vacuum, the velocity of light is constant and always exceeds that of the emitter. It enters the formulae determining the radiation, as a universal constant. Radiation in a vacuum is therefore determined solely by the nature of the emitter and the law of its motion. The case is different in a refractive medium. The phase and group velocities of light differ from those in a vacuum. They depend on the properties of the medium and on the frequency of the light. In optically anisotropic media, they are a function of the direction of propagation and polarization of the waves. In media of limited dimensions, changes in the velocity of light during transition through the boundary of the media are also of importance. Hence, in a refractive medium, the ratio between the velocity of the emitter and that of wave propagation depends considerably on the velocity of light in a medium and on its changes. Unlike a vacuum, the ratio may, notably, exceed unity. As a result, not only radiation properties but sometimes even the very fact of its origination depend on the peculiarities of light propagation in a medium. The Vavilov-Cerenkov effect is a case in point.

Radiation in a medium naturally also depends to a very great extent on the nature of the emitter. The theory makes it possible to foretell the properties of the Vavilov-Cerenkov radiation not only for a moving electric charge, but also for other cases. For instance, similar to an electric charge, the Vavilov-Čerenkov radiation should have also been produced by a magnetic charge, had it been proved to exist[7].

Whereas the question of radiation of a magnetic charge should now, too, be considered as being only theoretically possible the question of the Vavilov-Čerenkov effect for magnetic and electric dipoles and multipoles is quite real at present.

As a matter of fact, analysis of the radiation of a moving system of particles may prove necessary in resolving the numerous tasks related to processes in plasma and to problems of acceleration of particles. It is evident that a system of particles may, notably, be quasi-neutral, but it may possess an electric and, particularly, a magnetic moment due to moving ring currents.

A system of particles may not only move as a whole; it may also have natural frequencies of oscillations. This is true to an even greater extent of such systems as a moving atom, ion or atomic nucleus. An electron moving in a magnetic field may likewise possess natural frequency (Larmor frequency of revolution about the lines of a field). Therefore, apart from generalizing the theory of the Vavilov-Cerenkov effect, analysis is also required of the general case of radiation of systems possessing natural frequencies of oscillations[5].

Such a general analysis also includes the Vavilov-Cerenkov effect. The latter corresponds to the limiting case when the natural frequency is zero.

The fact that the theory of radiation of a charge moving with a velocity exceeding that of light has not been revised in the past twenty years does not mean at all the theory of this effect has been fully consummated. This can be seen from the following example. L. I. Mandelstam was the first to point out that it is not necessary for a charge to move in a continuous medium in order to radiate during super-light velocity*. The radiation re-

* See article by V. L. Ginzburg and I. M. Frank[9].

mains the same if the charge moves along the axis of a hollow cylindrical channel inside the medium, provided the diameter of the channel is small in comparison with the length of the emitted wave. For practical purposes this is very important, since it makes it possible to obtain radiation in a medium under conditions when the emitter does not collide directly with the atoms of the medium, which may deform or destroy it. It seemed that this applied also to the radiation of a dipole in a medium.

As recently shown, however, by V. L. Ginzburg and his associates, this question is not so simple as it appeared before[10]. The properties of a medium directly adjacent to the dipole may play an important part, and the presence of a channel of any, even the smallest, diameter cannot, therefore, be ignored.

This important factor has called for a critical analysis of the formerly obtained data as well. Thus, two contradictory results were obtained by two different methods for the radiation of a magnetic dipole[4,7]. It may now be assumed that this was not due to the erroneousness of one of the methods used, but to the fact that they differently took into account the effect of the medium adjacent to the moving dipole. Possibly both results are correct, but they apply to different physical cases. The matter requires, however, further consideration.

The series of problems dealt with in this paper, despite their diversity, comprises but the simplest case of radiation in a medium, namely radiation during which the translational motion of the system may be regarded as uniform and rectilinear.

Transition radiation

A typical example of radiation in a medium and, notably, during the uniform motion of an electric charge, is provided by the so-called transition radiation. The assertion that there is no radiation during a rectilinear and uniform motion of an electric charge at a velocity smaller than the phase velocity of light is correct only under the condition that the velocity of light along the path of the particle remains unchanged. For example, if a uniformly moving charged particle crosses the boundary of two media with different indices of refraction, there appears transition radiation. Radiation appears because the jump which the magnitude of the phase velocity of light undergoes at the boundary of two media is to some extent equivalent to the jump in the magnitude of the velocity of a particle. The equivalence to bremsstrahlung becomes complete in an extreme case, when the particle moves from vacuum to a metal in which light is absorbed over a length smaller than the wavelength of the light. The intensity of the transition radiation is at its maximum in this case. In the optical range of the spectrum - the only region in which transition radiation occurs - the spectrum and magnitude of the radiated energy are then exactly identical to those of the radiation which would have been produced by an electric charge and a charge of the opposite sign, moving towards it (its electric image in the metal), and which stop instantaneously at the point of encounter.

The intensity of transition radiation at low velocities is proportional to the kinetic energy of the particle, and it rises in the region of relativistic velocities as the logarithm of the total energy. Like bremsstrahlung, it becomes sharply directed in this case. It has been suggested that transition radiation might be useful in determining the energy of ultra-relativistic particles. This is important because it is very difficult to use for this purpose the Vavilov-Cerenkov effect for ultra-relativistic particles. As is well known, the angle at which the Vavilov-Cerenkov radiation is directed, and its intensity, attain in this case a practically constant value.

The use of transition radiation is, however, impeded by the fact that its intensity is very low. The probability of emission of a photon is of the order of the fine structure constant, i.e. of the order of a hundredth. If it is not possible to sum up transition radiation from many plates, observation of an individual particle by transition radiation may be carried out with but little efficiency. In this connection we should like to note the peculiarities of transition radiation at ultra-relativistic velocities. Unlike particles with a low velocity, transition radiation is almost the same during the incidence of such a particle from vacuum on a transparent dielectric as during the incidence on a metal. This is easy to understand by analogy with bremsstrahlung. Indeed, a change in the velocity of light is equivalent to a slight change in the velocity of the particle. But even a small change in the velocity of an ultra-relativistic particle means a great change in its energy, i.e. great deceleration of the particle. This peculiarity may permit the summing of transition radiation from the surfaces of many parallel transparent plates in a vacuum.

The second peculiarity consists in the fact that at ultra-relativistic velocities, the equilibrium field entrained by the particle in a vacuum is formed along a considerable path length. Consequently, to prevent the intensity of radiation from being reduced, the vacuum layers between the plates should not be less than some preset magnitude. For instance, for a proton with energy of 10^{11} electronvolts, this minimum distance is of the order of 1 mm, which is reasonable; but for a proton with energy of 10^{14} electronvolts it rises to the unreasonable magnitude of a kilometer.

I have dwelt on the subject of transition radiation in order to emphasize the peculiarity of the optical phenomena for radiation sources moving in refractive media, which so greatly depends on the peculiarities of propagation of light in a substance.

It should be noted that although the theory of transition radiation was developed by Ginzburg and the author of this lecture[11] more than ten years ago, and has since been analysed in a number of works* it has not yet been studied experimentally. The situation in this case is almost the same as in the case of the Vavilov-Cerenkov radiation before their papers were published. There is no doubt that transition radiation has also been observed on numerous occasions by various physicists, since the glow of the surfaces of electrodes under the impact of bombarding particles is well known. But even today the part played in this glow by luminescence, bremsstrahlung, and transition radiation has not been elucidated. The most reliable data on transition radiation have recently been obtained by A. E. Chudakov. Using the coincidence method, he observed photons emitted from the surface of a metal foil during the incidence on it of fast electrons from radiophosphorus. The intensity of radiation thus found proved to coincide with the estimated intensity for transition radiation, at least in the order of magnitude**.

It is also worth mentioning that transition radiation is practically always an intrinsic part of the Vavilov-Cerenkov radiation due to the limited thickness of the radiator. As shown by V. E. Pafomov for a radiator of very small thickness this factor should be taken into account[16].

* See, for instance, the papers by Garibyan and Pafomov and the references cited therein[12].

** In the book by Jelley, *Čerenkov Radiation*[18], with which I had the opportunity of becoming acquainted after this paper had been written, there is mention of the fact that in 1958 the author together with Elliott and Goldsmith observed a radiation emitted by 1.5 MeV protons incident on a polished aluminium target. On basis of the data on the intensity and polarization, the investigators concluded that the glow was transition radiation.

Radiation spectrum and quantum interpretation of the phenomenon

The radiation of a charged particle uniformly moving at a velocity exceeding that of light may, as is well known, be fully described by the methods of classical electrodynamics. The quantum theory of this phenomenon was first developed by Ginzburg[3] and then by many other investigators*. Ginzburg has shown that the classical formula for the cosine of the angle at which radiation occurs is correct up to a very small correction of the order of magnitude of the ratio between the energy of the radiated photon and the total energy of the moving emitter. (Even for an electron the ratio is less than 10^{-5}.) If this slight quantum correction contained in the exact formula is disregarded, identical relations between the frequency of the radiated light and the direction of its emission are obtained by both the classical and the quantum methods. Let us write them down in a quantum form, for a system possessing a natural frequency ω_0[14,5], where ω_0 is the frequency in the laboratory system of coordinates, that is, $\omega_0 = \omega_0' \sqrt{1 - \beta^2}$.

There is no necessity of assuming in this case that ω_0 is the only natural frequency possessed by the system. It may be regarded as a component of a complex spectrum of frequencies and it should be sufficient to study the radiation related to this frequency.

If the momentum of the photon, which in a medium should be assumed to equal $n\hbar\omega/c$, is very small in comparison with that of the emitter, then the law of momentum conservation during radiation may be expressed as follows

$$\frac{n\hbar\omega}{c}\cos\Theta = \frac{\Delta E}{v} \qquad (1)$$

where ΔE is the change in the kinetic energy of the emitter, and v is its velocity. From their ratio we obtain the magnitude of the change in the momentum of the system.

The change in kinetic energy is apparently determined by the energy of the radiated photon $\hbar\omega$ and the change in the internal energy of system $\hbar\omega_0$

$$\Delta E = \hbar\omega \pm \hbar\omega_0 \qquad (2)$$

The term $\hbar\omega_0$ should be taken with a minus sign if, when emitting the pho-

* See, for example, review[8].

ton, the system passes from an upper energy level to a lower one, that is, if the energy of the emitted photon is supplied, partly at least, from excitation energy. The plus sign should be used if the system becomes excited in the process of emission, i.e. if the kinetic energy is spent both on radiation and excitation.

By combining Eqs. (1) and (2), we obtain

$$\frac{n\omega}{c}\cos\Theta = \frac{\omega \pm \omega_o}{v} \tag{3}$$

Factor $\hbar$ has been cancelled out and the equation does not, indeed, contain anything of a specifically quantum nature. The same result is also obtained from classical wave analysis.

In Eq. (3) we can distinguish three cases:

Case 1 - Let us assume that

$$\frac{n\,v}{c}\cos\Theta = 1 \tag{4}$$

Then Eq. (3) is satisfied only if $\omega_o = 0$. This is precisely a case of the Vavilov-Cerenkov radiation, while (4) is a well-known condition determining the direction of emission of light for this radiation. The natural frequency $\omega_o = 0$ required for bringing into effect (4) means that the moving system should contain a source of a time-independent electromagnetic field (an electric charge, a constant dipole moment, etc.). Consequently, for the Vavilov-Cerenkov radiation to take place it is necessary that the constant component of the field should differ from zero. In this case Eq. (4) yields the relation between angle Θ and the radiated frequency, inasmuch as the refraction index $n(\omega)$ is a function of frequency.

Case 2 - Suppose now that the left-hand side of Eq. (4) is less than unity. Then Eq. (3) may be satisfied only if ω_o has a minus sign, i.e.

$$\frac{n\omega}{c}\cos\Theta = \frac{\omega - \omega_o}{v} \qquad \frac{vn}{c}\cos\Theta < 1 \tag{5}$$

This is nothing else but the Doppler condition for a source of light, moving in a medium. It has already been obtained by Lorentz when studying the optics of moving media.

Eq. (5) may evidently be expressed in the following ordinary way

$$\omega = \frac{\omega_0}{1 - \frac{vn}{c}\cos\Theta} \tag{5a}$$

It determines the frequency when the component of the velocity along a ray, $v \cos \Theta$, is less than the phase velocity of light c/n for frequency ω.

Eqs. (5) or (5a) differ from the usual Doppler condition for a source of light moving in a vacuum only in that the velocity of light in a vacuum has been replaced by the phase velocity c/n. If v is small in comparison with the phase velocity of light, and the dispersion of light is not great in the range of frequencies close to ω_0, this does not lead to anything fundamentally new. There is only a change in the absolute magnitude of the Doppler shift. It is so obtained as if it had been in a vacuum for a velocity equal to nv, i.e. n times greater. If the dispersion of light in the medium is great, there arise important peculiarities. The presence of dispersion should not be ignored in any medium when the velocities of motion are comparable to the phase velocity of light. Indeed, with n = constant and for angle $\Theta = 0$, the quantity $(vn/c) \cos \Theta$ would tend to unity with an increase in v while ω, as can be seen from (5a), would tend to infinity. At still greater velocities, the inequality sign in (5) would not be valid and, consequently, (5) would have no solution. As a matter of fact, the refraction index of any media becomes, practically equal to unity at sufficiently large values of ω. Hence, the Doppler frequency in this case is the same as it would have been in vacuum, i.e. it is certainly finite. In other words, at any velocity v and any value of Θ, Eq. (5) will have a solution. Moreover, as will be shown below, there may be not one but several solutions[4,5] ("complex" Doppler effect).

Case 3 - The third case takes place when the left-hand side of Eq. (4) is greater than unity. Then a plus sign should be before ω_0 in Eq. (3), and thus

$$\frac{n\omega}{c}\cos\Theta = \frac{\omega + \omega_0}{v} \qquad \frac{vn}{c}\cos\Theta > 1$$

This is a generalization of Doppler's formula for the case[4,14] when the velocity of the emitter exceeds the phase velocity of light for a radiated frequency*. It determines the « super-light » Doppler frequencies. Like the Vavilov-Čerenkov effect, the super-light Doppler frequencies appear when the velocity exceeds some threshold velocity. They are radiated simultaneously with ordinary frequencies, but only at sufficiently high velocities and within some range of acute angles.

It can be seen from the above quantum analysis that the plus sign at ω_0 in (2) and (6) respectively means excitation of the system. Hence, radiation of super-light photons occurs not during the transition from the upper, i.e. excited state into the lower, as in a general case, but quite the contrary, from the lower into the upper state, the energy being supplied from the kinetic energy of the translational motion of the system[14]. Such a radiation, accompanied by excitation of the system should take place spontaneously if the system is in the lower energy state. This is likewise possible as a spontaneous transition of the system from the upper energy state into the lower, accompanied by emission of photons with a frequency satisfying (5). As a matter of fact, the transition occurs in either case between the same energy states, and the question as to which of them takes place spontaneously is wholly determined by the initial state and the requirements of the conservation laws. In this case Eqs. (5) and (6) are in equal degree consequences of these laws.

The question regarding Doppler's effect in a refractive medium may also be considered within the framework of classical physics. From the viewpoint of classical physics, the results are interpreted as follows. Oscillations with natural frequency ω_0 bring about the appearance of radiation with frequencies which depend on the direction of propagation. It forms a spectrum of Doppler frequencies, which may be of two types. There is always a spectrum of radiation with frequencies satisfying Eq. (5), whose reaction on the emitter causes its damping. Under certain conditions, another spectrum with frequencies meeting Eq. (6) appears in addition to the first. The reaction of radiation of these frequencies promotes the building-up of oscillations. If damping prevails over building-up, oscillations will not arise by themselves in a system for which the classical formulae are correct, and if they existed at the beginning, they will attenuate.

In a quantum system the situation is fundamentally different. The processes of quantum radiation should be considered separately for spectra of both

* Apparently Eq. (6) may be put down in a form similar to (5a). The difference consists only in that the sign in the denominator of the right-hand part of Eq. (5a) should be changed.

types. Therefore, if a process corresponding to Eq. (6) is possible, it is certain to take place, i.e. the system will become excited owing to its own kinetic energy, radiate light, and pass in the usual way to the lower state. In principle, a two-photon mechanism is also possible, photons of both types being radiated simultaneously. Hence, as in the Vavilov-Čerenkov effect, a system possessing a natural frequency of oscillations will spend its kinetic energy on radiation at super-light velocity[14,15].

This can be formulated in the following way: as is well known, motion at a velocity greater than that of light is impossible in a vacuum. It is possible in a medium, but Nature does not lift its ban completely. Any system capable of interacting with radiation will slow down at a super-light velocity by radiating light.

Radiation thresholds

It is evident from the above analysis that the radiation spectrum is determined by the velocity of motion of the system v, its natural frequency ω_0 and the phase velocity of light c/n in a medium in which the radiation is emitted. Both the Vavilov-Cerenkov effect and the Doppler super-light effect are possible, as can be seen from (4) and (6) if $vn(\omega)/c > 1$. This obvious condition for the threshold of their appearance means that the velocity of motion should exceed the phase velocity of light.

This statement, correct for an isotropic medium, determines the threshold of emission of light of a given frequency ω for which the refraction index equals $n(\omega)$. As the refraction index depends on frequency, the threshold is different for another ω. This justifies raising the question in another way: under what condition do the Vavilov-Čerenkov effect and Doppler super-light effect generally become possible in a given medium?*

During radiation in a medium there is yet another peculiarity which likewise appears under certain threshold conditions. It consists in the following.

* For the Vavilov-Čerenkov radiation in an isotropic medium, this point regarding the threshold is elementary, since the latter is determined simply by the maximum value assumed by the refraction index in the given medium. Of importance for further consideration is the fact that for a frequency corresponding to n_{max}, the phase and group velocities are equal (see Eq. (10)), it being evident that for n_{max}, $dn/d\omega = 0$. Hence the fact that the threshold velocity of motion is equal to the phase velocity means that it is also equal to the group velocity of light.

Eq. (3) and, naturally, its sequels (4), (5), and (6), are not linear with respect to ω. As a matter of fact, they contain the refraction index $n(\omega)$ which is a function of the radiated frequency. As a result, not one but several values of ω, satisfying (3) are possible in some cases for given values of Q, v, and ω_0. This means that several components of different frequency may be radiated simultaneously in a given direction. The appearance of such additional frequencies, i.e. of the so-called complex effects of radiation, is possible only under certain conditions. They may arise not only in the superlight Doppler effect and Vavilov-Čerenkov's radiation, but also in the ordinary Doppler effect subordinated to Eq. (5).

L. I. Mandelstam was the first to draw attention to the fact that the condition under which the complex Doppler effect appeared[4] was related to the magnitude of the group velocity of light. The statement proved to be of a general natures.

If we consider radiation in the direction of motion, then in all the enumerated cases the condition for appearance of the radiation or of its new components is that the velocity of the emitter should equal the group velocity of light for a frequency which may radiate (i.e. which satisfies condition (3)). This threshold frequency should evidently satisfy Eqs. (4), (5), or (6), depending on the kind of radiation under consideration.

It is well known that in a refractive medium the transfer of radiation en-

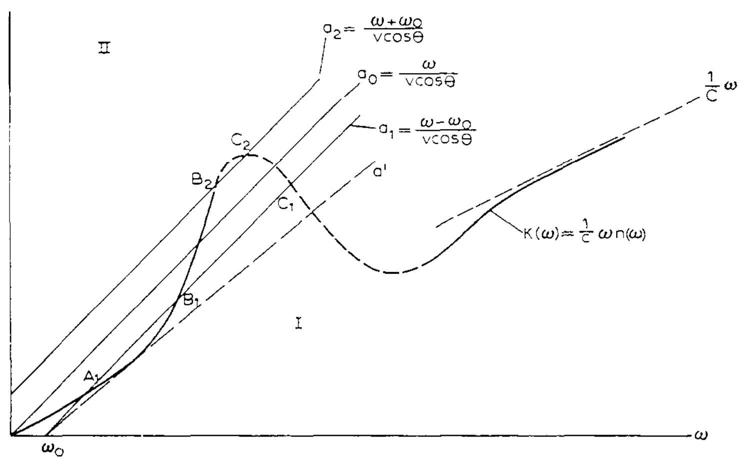

Fig. 1.

ergy occurs not with the phase but precisely with the group velocity. Thus it is not surprising that the group velocity of light is of importance for the processes of radiation in a medium.

The fact that the radiation threshold is connected precisely with the group velocity can be explained by some simple qualitative considerations. Let us assume that the conditions for appearance of the radiation have been fulfilled. Radiation arises and carries energy away from the emitter. Suppose furthermore, that the velocity of motion changes, and approaches the threshold velocity. When the threshold is attained, the radiation should disappear, i.e. removal of energy from the emitter ceases. When the velocity of motion equals the group velocity of light, this will actually take place, since there occurs simply a transfer of energy together with the emitter.

The condition of appearance of the complex effect may be easily determined by analysing the chart in Fig. 1. The curve in Fig. 1 represents dependence of the magnitude of wave vector $k(\omega) = \omega n(\omega)/c$ on the frequency for some imaginable medium. In addition to curve $k(\omega)$, Fig. 1 contains three straight lines whose equations are

$$a_0 = \frac{\omega}{v \cos \Theta} \tag{7}$$

$$a_1 = \frac{\omega - \omega_0}{v \cos \Theta} \tag{8}$$

$$a_2 = \frac{\omega + \omega_0}{v \cos \Theta} \tag{9}$$

The points where the straight lines cross the curve seem to determine at once the frequencies satisfying Eqs. (4), (5), and (6) respectively.

The tangent of the angle of incline of the straight lines a_0, a_1, a_2 to axis ω apparently equals $1/v \cos \Theta$. Let us assume, in accordance with Fig. 1, that $\cos \Theta \geqslant 0$, that is, $\Theta \leqslant \pi/2$.

The nature of intersection of the straight lines a with curve $k(\omega)$ may differ. If we move along the straight line in the direction of increased ω, the straight line may go over at the point of intersection from the region underlying the curve (Region I) into the region above the curve (Region II). This takes place if the slope of the tangent to curve $k(\omega)$, i.e. $dk/d\omega$, is less than $\gamma = 1/v \cos \Theta$ (see, for example, point A, on the straight line a_1).

On the contrary, if $dk/d\omega > 1/v \cos\Theta$ then there is a transfer from Region II into Region I at the point of intersection. Finally, $dk/d\omega = 1/v \cos\Theta$ takes place at the point of tangency.

As can be easily proved, the slope of the tangent to curve $k(\varpi)$ is equal to the reciprocal of the group velocity of light. Indeed, when there is no absorption, the group velocity W, as is well known, satisfies the relationship

$$\frac{1}{W} = \frac{dk}{d\omega} = \frac{1}{c}\frac{d}{d\omega}(\omega n) = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right) \qquad (10)$$

Hence, the group velocity of light for frequencies which can be radiated is related to the velocity of motion v and $\cos\Theta$ by the relationships*

$$\frac{v\cos\Theta}{W(\omega)} < 1 \qquad \text{transition from I into II} \qquad (11)$$

$$\frac{v\cos\Theta}{W(\omega)} > 1 \qquad \text{transition from II into I} \qquad (12)$$

$$\frac{v\cos\Theta}{W(\omega)} = 1 \qquad \text{tangency} \qquad (13)$$

At sufficiently large ϖ, the quantity W becomes equal to c. Indeed, the refraction index tends to unity, and hence curve $k(\omega) = \omega n/c$ approaches a straight line with a slope of $1/c$.

The straight lines a rise more abruptly since $v < c$ and consequently

$$\frac{1}{v\cos\Theta} > \frac{1}{c}$$

Hence, all the three straight lines a at great ω are in Region II.

This entails a number of consequences. First of all, it is evident that the straight line a_1 will necessarily cross curve $k(\omega)$, i.e. Eq. (5), as has already

* The magnitude determined by (10) has the meaning of the group velocity of light only when there is no strong absorption, i.e. in those regions of the spectrum for which the medium is transparent. The part of curve $k(\omega)$ corresponding to the region of anomalous dispersion, in which there is unquestionable dispersion, is shown in Fig. 1 by a dotted line. The peculiarities of radiation for frequencies getting into this region call for special consideration.

been noted, must always have a solution. As a matter of fact, the straight line a, passes through point $\omega = \omega_0$ lying on the axis of abscissae, which means that the straight line must go over somewhere from Region I into Region II. Moreover, it means that at any rate a frequency is radiated for which inequality (II) corresponding to a transition from I into II is applicable.

The straight lines a_0 and a_2 as might have been expected, do not always cross curve $k(\omega)$. This requires that their incline to the abscissa axis should be sufficiently small. This means that the velocity should be high and angle Q should not be large.

At great ω both these straight lines also prove to be in Region II. It follows from this that if there are crossings then at any rate, the last of them which determines the highest of the radiated frequencies corresponds to a transition from Region I into Region II. The result is then again that there is a frequency in the radiation, for which inequality (II) is valid. For forward radiation, i.e. Q = o, this means that there is a component for which v < *W* and, consequently, for at least a part of the radiation, energy is propagated at a higher velocity than that of the source of lights.

It also follows from the above that if there is a frequency satisfying condition **(12)** (for instance corresponding to point B_1 on the straight line a,), the composition of the radiation will infallibly be complex, since there must be a frequency or frequencies satisfying condition (II). (In the general case the number of possible crossings for the straight line a, is always odd, and for the straight line a_2 always even.)

The boundary of the appearance of radiation or of new components of radiation is evidently represented by a case when the corresponding straight line a begins to touch curve k(o). This means the fulfilment of Eq. (13). With Q = o we obtain, in agreement with the above, v = *W* for the threshold frequency.

The dotted line a' in Fig. 1 corresponds to the threshold of appearance of the complex effect for the ordinary Doppler effect. As seen from the figure, the frequency begins to split when the slope of the straight line a_1 increases in comparison with that of the dotted line. This means that the complex Doppler effect arises in this case not when the velocity increases in comparison with the threshold velocity, but, quite the contrary, when it decreases or when the angle becomes larger (it is worth recalling that the tangent of the incline of the straight line a, equals $1/v \cos \Theta$). This is explained by the fact that the complex Doppler effect takes place here only within some range of velocities or angles, and the dotted line corresponds to the upper, and not the lower threshold of the effect.

It has been assumed up till now that angle Q is acute, i.e. that the product v cos Q is positive. What was said above regarding the complex Doppler effect may also be applied to the case of obtuse angles Q but in this case negative group velocity will have to be taken into consideration. It appears that the threshold for the appearance of the complex Doppler effect, with $Q > \pi/2$ is determined by Eq. (13). Quantity cos Q is negative in this case, therefore Eq. (13) is valid only when the quantity *W* is less than zero. The import of negative group velocity for the Vavilov-Čerenkov effect was first investigated by Pafomov[16,12b] who pointed out that such a case should be real in anisotropic media*. This is a very interesting case. We are accustomed to the idea that the Vavilov-Čerenkov radiation is directed forward at an acute angle. This is, however, correct only if the group velocity is positive. If it is negative, the picture is quite different.

Fig. 2a shows schematically the ordinary case of the Vavilov-Cerenkov radiation. The phase velocity for radiated light $u = c/n$ forms in this case an

* This is related to the fact that in an anisotropic medium the direction of the group velocity does not coincide with that of the phase velocity. This question is treated in the next section of the lecture.

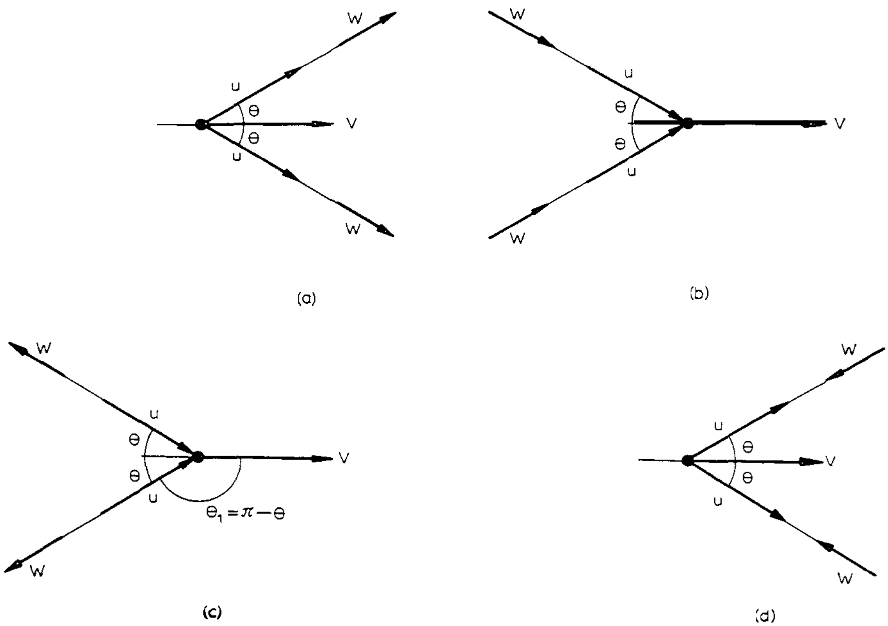

Fig. 2.

acute angle Θ with the direction of velocity v. The equation of electrodynamics also permits of the solution schematically represented in Fig. 2b. The direction of phase velocity, i.e. the direction of wave propagation forms in this case, too, the same acute angle Θ with a velocity vector. The waves do not, however, come from the emitter, but towards it. The first case is interpreted as a radiation of waves, and the second as their absorption. If there is no source of energy feeding the waves, flowing to the emitter, then the case of Fig. 2b is not realizable and the corresponding solution is rejected. But this is correct only if the group velocity is positive, i.e. if its direction coincides with that of phase velocity (see vector *W* in Figs. 2a and 2b). The direction of the energy flux coincides in this case with the direction of phase velocity and, consequently 2a really corresponds to the radiation of the waves, and 2b to their absorption. In a medium with a negative group velocity, vector *W* is so directed as to meet vector *u* (the medium is considered optically isotropic, and hence vectors *u* and *W* may be only parallel or anti-parallel). Therefore, with $W < 0$, Fig. 2c corresponds to radiation of energy, and 2d to its absorption. Hence, if the group velocity is negative, the direction of the energy flux of the Vavilov-Čerenkov radiation forms an obtuse angle $\Theta_1 = \pi - \Theta$ with the direction of the velocity, and the motion of the waves is directed not from the particle, but, quite the contrary, towards it.* A similar analysis can also be made of an emitter with a natural frequency ω_0, moving in a medium with a negative group velocity[5,12b].

It can be seen from the above that many substantial peculiarities of radiation in a refractive medium are actually related not only to the magnitude of the phase velocity of light, but also to the group velocity of light. It may be expected that the role of the group velocity of light will reveal itself most distinctly in anisotropic media in which the directions *u* and *W* form some angle with one another.

Radiation in optically anisotropic media

Radiation of a light source moving in a crystal should possess a number of features as compared with that in isotropic media. Interest in this range of

* The analysis given in Fig. 2 is similar in many ways to the example given in L. I. Mandelstam's lectures on the refraction of light by a medium with a negative group velocity (*Collected Works* by L. I. Mandelstam, Vol. 5, p. 463).

problems has enhanced recently in connection with the studies of the processes in plasma*. As to propagation of waves, a plasma placed in a magnetic field is similar to a uniaxial gyrotropic crystal.

The Vavilov-Cerenkov effect in crystals was first investigated theoretically by V. L. Ginzburg[13] and then by other investigators (see, for example, review[8]). It has not, however, been studied experimentally to this day.

The equation determining the radiated frequency ω remains the same as in an isotropic medium, i.e., ω is determined by Eq. (4). The magnitude of the refraction index n in the case of an anisotropic medium depends, however, not only on the frequency of light, but also on the angle and polarization. The result is that for the Vavilov-Cerenkov radiation the cone of normals to the wave surfaces is not circular in this case, as in an isotropic medium, but may have quite an odd shape. Thus the direction of velocity does not necessarily coincide with the axis of the cone, and in some cases may even lie beyond the cone[16].

Another peculiarity is related to polarization of the light. The Vavilov-Čerenkov radiation is always polarized. As a rule, polarization of the light in this phenomenon does not attract attention, since it has not been used so far in present-day practical application of the radiation. However, from the viewpoint of the mechanism of the phenomenon, polarization is highly important. It is worth mentioning, for example, that the radiation of a magnetic charge, if it exists at all, could be distinguished at once from the radiation of an electric charge, since in this case the magnetic and electric vectors change places. The question of polarization of light is also of importance for the quite real case of radiation of dipoles and multipoles, though it has not yet been studied experimentally.

The role of polarization is manifested most distinctly in an anisotropic medium. First of all, one can obtain here, depending on the polarization of the radiated light, not one, but two cones of wave normals corresponding to so-called ordinary and extraordinary rays in a uniaxial crystal. Moreover, the distribution of the radiation intensity is a complex function of the angles and is related to polarization of the light. The fulfilment of condition (4) does not suffice to bring about radiation, since the intensity of the waves of a given polarization may prove to equal zero. For example, if a particle moves in the direction of the axis of a uniaxial crystal, the cone of ordinary rays must disappear in the radiation[8].

* Some of the problems connected with plasma are dealt with in I. E. Tamm's Nobel Lecture, see this book, p. 470.

The third peculiarity is related to the fact that in an anisotropic medium the direction of the ray, i.e. the direction of a narrow beam of light, does not, generally speaking, coincide with the normal to the wave surface. There exist such directions of rays in a crystal, for which the normal to the wave surface forms some angle α with the ray (see Fig. 3).

The velocity at which the phase of the wave propagates in the direction of the ray, as can be seen from Fig. 3, is $1/\cos\alpha$ times greater than the phase velocity, i.e. $u' = u/\cos\alpha = c/n \cos\alpha$. We shall call u' the velocity of the waves along the ray. It should not be confused with the group velocity of light, i.e. with the velocity of transfer of light energy which, naturally enough, is also directed along the ray. The group velocity equals velocity u' only under the condition that there is no dispersion of light in the medium. Indeed, the velocity of the waves along the ray does not depend in this case on frequency, and hence the group of waves moves with the same velocity.

The velocity of the waves along the ray is important for radiation in anisotropic media. Let us consider in this connection the threshold velocity for the appearance of the Vavilov-Čerenkov effect. The assertion that the Vavilov-Čerenkov radiation for a light of frequency ω arises at a velocity greater than the phase velocity of light with the given frequency implies

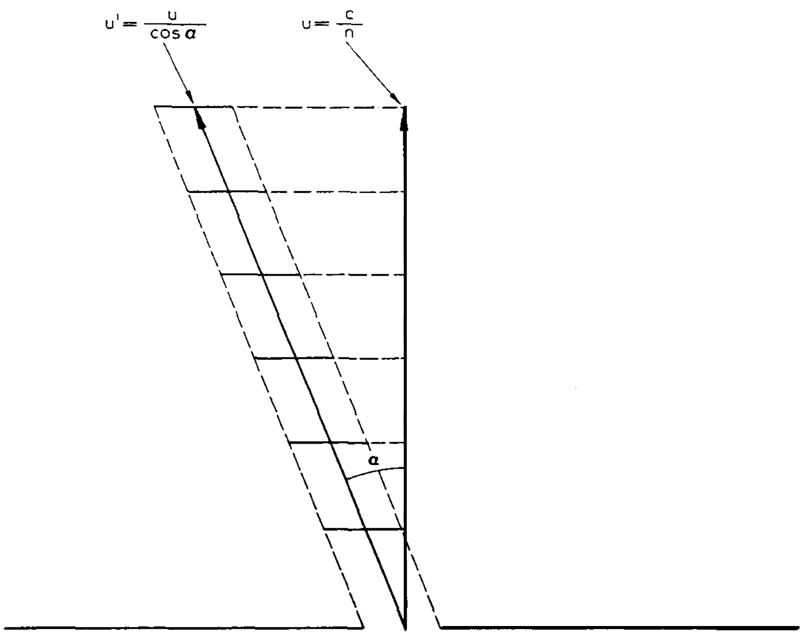

Fig. 3.

that the medium is isotropic. If this statement be considered applicable to anisotropic media (which, as will be seen below, is not always the case), it is necessary, at least, to indicate with which direction of the phase velocity the velocity of motion is to be compared.

Eq. (4), i.e. (vn/c) cos Q = I, is also valid for anisotropic media, and in this case $c/n = u$ is the phase velocity for the given direction of the normal to the wave, forming angle Q with vector $\vec{v}$. As is well known, when the velocity approaches threshold velocity in an isotropic medium, angle Q decreases to zero, i.e., the cone of wave normals is compressed in the direction $\vec{v}$. In a crystal, the cone of wave normals is likewise compressed in this case towards some axis which, as a rule does not, however, coincide with $\vec{v}$. If this axis is represented by the direction of the velocity, the threshold Q = o and then we obtain from (4) that v, = c/n where c/n is assumed for the direction $\vec{u} = (\overrightarrow{c/n})$ coinciding with $\vec{v}$. Hence, $\vec{v} = \vec{u}$. This relationship actually proves to be correct for boundary velocity in the usual cases of motion in a uniaxial crystal parallel or perpendicular to the optical axis. It has not, however, been stressed that it cannot always be applied.

It may be shown that the general condition for the appearance of the Vavilov-Cerenkov radiation of frequency ω should be formulated in the following way. The threshold velocity of the source of light should equal the velocity of waves along the ray in the direction of motion. In other words, the threshold velocity $\vec{v} = \vec{u}'$. For the threshold velocity, the direction of the ray coincides with $\vec{v}$ and not the normal to the wave which forms an angle α with $\vec{v}$. Hence in the general case, the threshold value is Q = α

In a special case, when the direction of the ray coincides with the wave normal in an anisotropic medium, i.e., α = o; $\vec{u}' = \vec{u}$. Then we have 3 = $\vec{u}$ for the threshold velocity. Finally, in an isotropic medium, where the phase velocity of light c/n is the same in all directions, it is possible to go over from vectors to scalar quantities, which means that v = u. Hence the well-known statement that the velocity equal to the phase velocity of light is the threshold velocity, has a limited field of application. It is a special case of a more general condition.

The above is easy to explain by using the Huygens principle for plotting the wave surface of radiation. This procedure is still generally used at present to describe the Vavilov-Cerenkov effect in an elementary way, and at the time it was one of the guiding ideas in the creation of its theory. This method can be easily applied to the case of an anisotropic medium.

The Huygens principle is frequently used in crystal optics to explain the peculiarities of behaviour of the so-called extraordinary ray during the refraction of light. The wave surface is found by the Huygens principle as an envelope of the waves emitted from separate points. Whereas, however, for an isotropic medium a sphere of radius $r = (c/n)\ t$ is plotted around every point, where t is the time of movement of the waves, a crystal calls for a different approach. Of importance is the distance covered by the wave from a given point in the given direction of the ray. The distance equals the velocity of the waves along the ray, multiplied by time t, i.e. $\vec{u}'t$. Therefore, the unknown quantity is represented by the envelope of the so-called surfaces of the rays plotted around every source of waves and determined by the equation $\vec{r} = \vec{u}'t$.

Let us apply the Huygens principle to the case of Vavilov-Čerenkov radiation in a uniaxial crystal. The velocity of the ordinary and extraordinary rays is not the same here and, therefore, generally speaking, two cones of waves are obtained. In order not to encumber the drawing, they are shown on separate Figs. 4 and 5. We have to consider each point of the particle trajectory as a source of waves. In this case the wave phase is determined by the instant of passage of the particle through a given point. Let us assume that at moment $t = -t_3$ the emitter was at point A_3, at moment $t = -t_2$ at point A_2 at moment $t = -t_1$ at A_1, and, finally, at the moment of observation $t = 0$ at point A_0.

For ordinary rays, the velocity of the waves along the ray, as in an isotropic medium, is equal to the phase velocity of light c/n and does not depend on the direction. The surfaces of the rays are simply spheres whose radii for points A_3, A_2, A_1 and A_0 are $(c/n)\ t_3$, $(c/n)\ t_2$, $(c/n)\ t$, and 0 respectively (see Fig. 4). The envelope of these spheres evidently represents a cone of

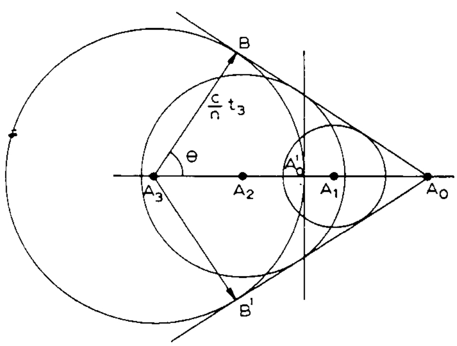

Fig. 4.

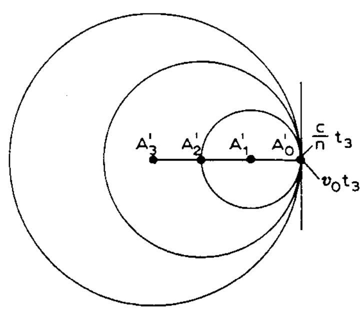

Fig. 4a.

circular cross section with the apex at A_0*. Its generatrices lying in the plane of the drawing are A_0B and A_0B'.

According to the Huygens principle, the directions of the rays are defined by the radius vectors drawn from some centre of the waves to the point of tangency with the envelope. For example, in Fig. 4 it is A_3B or A_3B' coinciding with the generatrices of the wave normal cone for ordinary rays. Thus the radiation cone is obtained for ordinary rays in the same way as in the Vavilov-Čerenkov effect in an isotropic medium. The substantial difference from an isotropic medium is related to the polarization of light and the distribution of intensity, depending on it. This was not taken into account in the construction.

From Fig. 4 it is not difficult to determine the magnitude of the threshold velocity. When the velocity diminishes, the distances between points A decrease. The threshold case arises when point A_0 occupies the position of A_0' on the surface of the sphere. (This case is depicted separately in Fig. 4a.) At lower velocities, one of the spheres lies completely within the other and they do not have a common envelope. In the threshold case, they have only a common point of tangency A_0'. Thus evidently $(c/n)\ t_3 = v_0 t_3$, i.e. $v_0 = c/n$. The cone of wave normals is compressed in the direction of velocity v, and the wave cone transforms into a plane perpendicular to the axis of motion at point A_0' (Fig. 4a).

The Huygens principle can also be applied in a similar way to obtain a wave cone for the extraordinary rays (Fig. 5). The difference consists in that surfaces of rays $\vec{u}'t_3, \vec{u}'t_2$, and $\vec{u}'t$, instead of spheres are plotted around points A_3, A_2, and A_1. The cone enveloping the surfaces with an apex at A_0 is not circular in the case shown in Fig. 5. The generatrices of this wave cone A_0C and A_0C' lie in the plane of the drawing. The lines perpendicular to them, for instance A_3D and A_3D' determine the wave normals, and their length is proportional to the phase velocities. The vectors drawn from A_3 to the points of tangency A_3F and A_3F' indicate the corresponding directions

* Strictly speaking, such an analysis presupposes that there is a superposition of monochromatic waves. Each point of the trajectory should, therefore, be regarded as a source of such waves emitted for an infinitely long time. Actually, it is only the summation of waves of various frequency that produces a light impulse when the particle passes through a given point. Hence, there exists, of course, not one, but an unlimited multitude of wave surfaces for a given frequency. The one that is generally plotted is singled out only by its passage through the instantaneous position of the particle (which we shall term the wave cone).

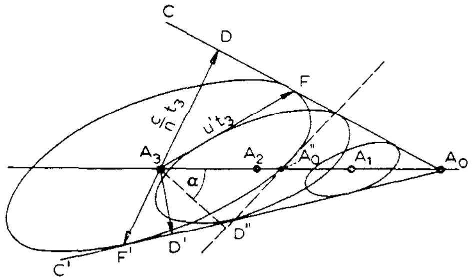

Fig. 5.

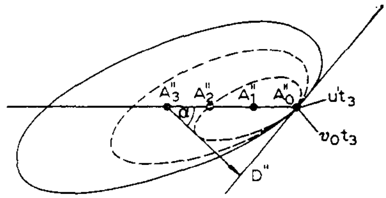

Fig. 5a.

of rays, which, as seen from Fig. 5, do not coincide with the wave normals. It can also be seen from the drawing that the direction of an extraordinary ray for the Vavilov-Cerenkov radiation in a crystal may even constitute an obtuse angle with the direction of velocity (direction A_3F' in Fig. 5).

It is not difficult to determine the magnitude of the threshold velocity for the appearance of extraordinary rays in the Vavilov-Čerenkov radiation, which, generally speaking, differs from threshold velocity for ordinary rays. The threshold case occurs when the velocity diminishes to such an extent that point A, coincides with point A_0''. In this case all the surfaces of the rays lie within each other and have a common point of tangency A_0''. It can be seen from Fig. 5 or 5a showing a threshold *case* that the threshold value is $v = v_0 = u'$. The wave cone then transforms into plane $A_0''D''$ and the wave normal forms an angle α with direction v. By tracing what happens to the cone of wave normals (its generatrices are A_3D and A_3D' in Fig. 5) during a decrease in velocity, i.e., when point A,, approaches A_0'', it is not difficult to prove that it is compressed not in direction v but in direction A_3D''. Hence, in a threshold case in Eq. (4), it may be assumed that not Q = o but $Q = \alpha$. Then Eq. (4) produces $(vn/c) \cos \alpha = 1$, i.e., actually $v = c/n \cos \alpha = u'$.

It is worth recalling that with the aid of Figs.4 and 5 we have determined the threshold of appearance of light of some given frequency ω. The velocity at which radiation generally appears is determined by a minimal magnitude of wave velocity of waves along the ray, namely $u' = u'_{min}$ in a given medium for a ray directed along the motion. For frequency ω' for which $u' = u'_{min}$ the velocity of the waves along the ray does not depend on frequency and is thus equal to the group velocity. Hence, we again come to the conclusion that the threshold is related to the group velocity.

The analysis of radiation of a system possessing a natural frequency of oscillations ω_0, may also be applied to the case of an optically anisotropic medium. The same peculiarities are manifested here as referred to in connection with the Vavilov-Čerenkov radiation. The connection between ω, Q, v, and ω_0 is determined, as before, by the same Eqs. (5) and (6) as in an isotropic medium, but now quantity n refers to the direction of a wave normal at an angle Q to the velocity.

The dependence of n on the direction leads to the fact that the connection between Q and the frequency of radiation ω at preset natural frequency ω_0 and velocity v is not elementary. To find Q, use can be made of the graphic method suggested by V. E. Pafomov[16] for analysing the Vavilov-Čerenkov effect in crystals, applying it to the case of an arbitrary ω_0 (see Fig. 6). The figure shows a section of a surface of wave vectors $\vec{k}(\omega) = (\overrightarrow{\omega n})/c$ for the given ω in the case of extraordinary rays in a uniaxial crystal. The surface indicating the dependence on the direction of vectors $\vec{k}$ (they are oriented along the normal to the wave) differs from that of refraction indices only by a constant factor ω/c (we consider magnitude ω as prescribed). Thus, for

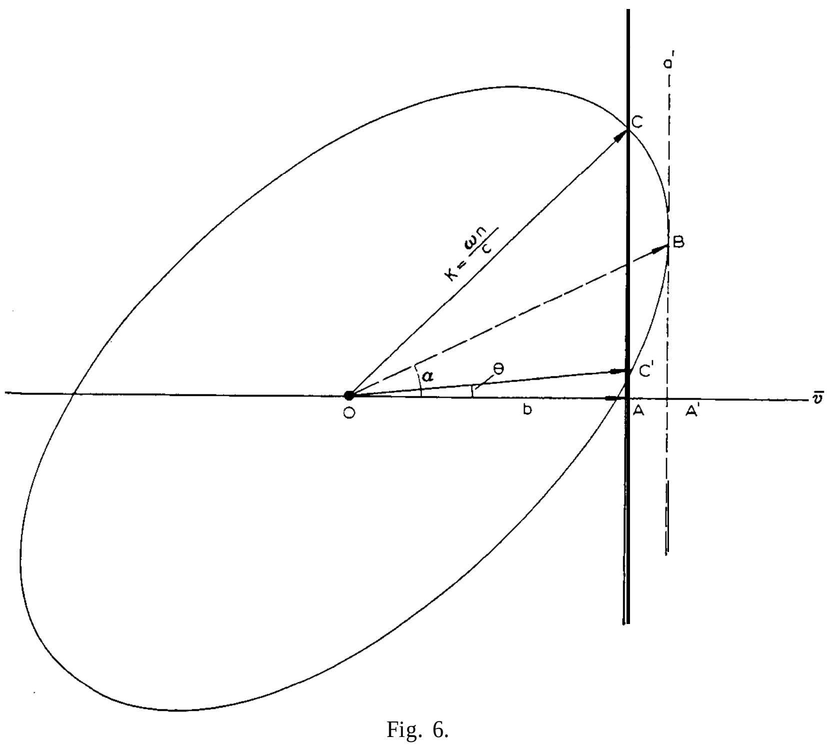

Fig. 6.

a uniaxial crystal, the surface represents an ellipsoid of rotation. Let us assume that axis $\vec{v}$ is the direction of motion of the emitter. Let us plot on axis $\vec{v}$ segment OA of length b which equals b_0, b_1, or b_2 depending on whether the analysis deals with the Vavilov-Cerenkov effect, the Doppler ordinary effect, or the Doppler super-light effect. Then

$$b_0 = \frac{\omega}{v} \tag{14}$$

$$b_1 = \frac{\omega - \omega_0}{v} \tag{15}$$

$$b_2 = \frac{\omega + \omega_0}{v} \tag{16}$$

At point A which is the end of b we shall plot plane a perpendicular to axis v. Let us consider the curve where the plane crosses surface $k(\omega)$ as a section of some cone with the apex at O. The generatrices of this cone OC and OC' lie in the plane of the figure. The cone defines the magnitude and direction of vectors $\vec{k}$ for light of frequency ω appearing in the case under consideration, i.e. for the given kind of radiation with preset ω_0 and v.

Indeed, as can be seen from Fig. 6, OA = b is a projection of vector OC or OC', *i.e.* of vector $k = \omega n(\omega, \Theta)/c$. Hence

$$\frac{\omega n(\omega, \Theta)}{c} \cos \Theta = b$$

By substituting the values of b from (14), (15), or (16) we obtain identical equations (4), (5), or (6).

It can be seen from Fig. 6 that the cone of wave normals may actually be not only asymmetric, but, as has already been mentioned, axis $\vec{v}$ may even lie outside the cone.

Plane a does not always cross the surface of $k(\omega)$. This corresponds to the evident fact that not every frequency is radiated for given v and ω_0. If b = b' = OA' (see Fig. 6), the plane touches the surface and, consequently b' = OA' is a boundary for the appearance of the given frequency ω in the spectrum. Vector k, i.e. the wave normal, coincides in this case with OB. It can be easily proved that it forms angle α with the direction of velocity, and with this, angle $\Theta = \alpha$ is inserted in Eq. (3), we obtain the following general condition for velocity v_o required for the appearance of frequency ω

$$\frac{\omega}{u'} = \frac{\omega \pm \omega_o}{v_o} \tag{18}$$

where u' is the velocity of the waves along axis $\vec{v}$ (positive or negative, i.e. directed along $\vec{v}$ or opposite to it). In a special case of the Vavilov-Cerenkov radiation, $\omega_o = 0$.

Radiation of a system possessing a natural frequency of oscillations and moving in an optically anisotropic medium was first studied by K. A. Barsukov and A. A. Kolomensky[17]. They elucidated a number of peculiarities of radiation related to the presence of ordinary and extraordinary rays and the significant role of wave polarization.

It is highly interesting that this seemingly more complex case appears to present even now some interest from an experimental point of view. Barsukov and Kolomensky made a special study of radiation of radio waves in the ionosphere which behaves like an optically anisotropic medium under the action of the earth's magnetic field. It is important that this medium possesses strong dispersion at some range of frequencies and that the complex Doppler effect is possible in it. Kolomensky and Barsukov have pointed out that this phenomenon may take place in the case of radio waves of appropriate frequency, transmitted by an artificial earth satellite moving in the ionosphere. They found that the Doppler shift of frequency of the order of ten to a hundred cycles per second should be accompanied in this case by splitting of the radiated frequency into components of several hundredths of a cycle per second apart. Apparently, with a well-stabilized frequency of the transmitter, such splitting could be detected.

I have aimed to prove in my lecture that there is a wide range of problems related to the radiation of sources of light, moving in refractive media. Radiation of an electric charge moving at super-light velocity in an isotropic medium, i.e. the experimentally investigated case of the Vavilov-Cerenkov effect, is, in essence, but a special, though a highly interesting instance in this realm of phenomena.

1. P. A. Čerenkov, *Dokl. Akad. Nauk SSSR,* 2 (1934) 451.
S. I. Vavilov, *Dokl. Akad. Nauk SSSR, 2* (1934) 457.
P. A. Čerenkov, *Tr. Fiz. Inst. Akad. Nauk SSSR im. P. N. Lebedeva,* 2, *No. 4. (1944)*
2. I. E. Tamm and I. M. Frank, *Dokl. Akad. Nauk SSSR,* 14 (1937) 107.
3. V. L. Ginzburg, *Zh. Eksperim. i Teor. Fiz.,* IO (1940) 589.
4. I. M. Frank, *Izv. Akad. Nauk SSSR, Ser. Fiz., 6 (1942) 3; J. Phys. USSR, 7 (1943) 49.*
5. I. M. Frank, *Zh. Eksperim. i Teor. Fiz., 36 (1959) 823 ; Joint Inst. Nucl. Res.,* P-237 (1958).
6. E. Fermi, *Phys. Rev., 57 (1940) 485.*
7. I. M. Frank, *Vavilov Memorial Symposium,* USSR Academy of Sciences Publishing House, Moscow, 1952.
8. B. M. Bolotovsky, *Usp. Fiz. Nauk, 62 (1957)* 201.
9. V. L. Ginzburg and I. M. Frank, *Dokl. Akad. Nauk SSSR,* 56 (1947) 699.
10. V. L. Ginzburg and V. Y. Eidman, *Zh. Eksperim. i Teor. Fiz.,* 35 (1958) 1509.
L. S. Bogdankevich, *Zh. Tekhn. Fiz., 29 (1959)* 1086.
11. V. L. Ginzburg and I. M. Frank, *Zh. Eksperim. i Teor. Fiz.,* 16 (1946) 15 ; *J. Phys.* USSR, 9 (1945) 353.
12. G. M. Garibyan, *Zh. Eksperim. i Teor. Fiz.,* 33 (1957) 1403.
V. E. Pafomov, *Zh. Eksperim. i Teor. Fiz.,* 36 (1959) 6.
13. V. L. Ginzburg, *Zh. Eksperim. i Teor. Fiz.,* 10 (1940) 608; *J. Phys. USSR,* 3 (1940) 101.
14. V. L. Ginzburg and I. M. Frank, *Dokl. Akad. Nauk SSSR,* 56 (1947) 583.
15. I. M. Frank, *Usp. Fiz. Nauk,* 30 (1946) 149.
16. V. E. Pafomov, *Thesis, Physical Insitute of the USSR Academy of Sciences, Moscow, 1957; Zh. Eksperim. i Teor. Fiz.,* 32 (1957) 366; 37 (1959) No. 6.
17. K. A. Barsukov and A. A. Kolomensky, *Zh. Tekhn. Fiz.,* 29 (1959) 954.
18. J. V. Jelley, *Čerenkov Radiation and its Applications,* Pergamon Press, 1958.