WOLFGANG PAULI

Exclusion principle and quantum mechanics

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The history of the discovery of the « exclusion principle », for which I have received the honor of the Nobel Prize award in the year 1945, goes back to my students days in Munich. While, in school in Vienna, I had already obtained some knowledge of classical physics and the then new Einstein relativity theory, it was at the University of Munich that I was introduced by Sommerfeld to the structure of the atom - somewhat strange from the point of view of classical physics. I was not spared the shock which every physicist, accustomed to the classical way of thinking, experienced when he came to know of Bohr's « basic postulate of quantum theory » for the first time. At that time there were two approaches to the difficult problems connected with the quantum of action. One was an effort to bring abstract order to the new ideas by looking for a key to translate classical mechanics and electrodynamics into quantum language which would form a logical generalization of these. This was the direction which was taken by Bohr's « correspondence principle ». Sommerfeld, however, preferred, in view of the difficulties which blocked the use of the concepts of kinematical models, a direct interpretation, as independent of models as possible, of the laws of spectra in terms of integral numbers, following, as Kepler once did in his investigation of the planetary system, an inner feeling for harmony. Both methods, which did not appear to me irreconcilable, influenced me. The series of whole numbers 2, 8, 18, 32... giving the lengths of the periods in the natural system of chemical elements, was zealously discussed in Munich, including the remark of the Swedish physicist, Rydberg, that these numbers are of the simple form $2n^2$, if n takes on all integer values. Sommerfeld tried especially to connect the number 8 and the number of corners of a cube.

A new phase of my scientific life began when I met Niels Bohr personally for the first time. This was in 1922, when he gave a series of guest lectures at Göttingen, in which he reported on his theoretical investigations on the Periodic System of Elements. I shall recall only briefly that the essential progress made by Bohr's considerations at that time was in explaining, by means of the spherically symmetric atomic model, the formation of the intermediate shells of the atom and the general properties of the rare earths. The question, as to why all electrons for an atom in its ground state were not bound in the innermost shell, had already been emphasized by Bohr as a fundamental problem in his earlier works. In his Göttingen lectures he treated particularly the closing of this innermost K-shell in the helium atom and its essential connection with the two non-combining spectra of helium, the ortho- and para-helium spectra. However, no convincing explanation for this phenomenon could be given on the basis of classical mechanics. It made a strong impression on me that Bohr at that time and in later discussions was looking for a *general* explanation which should hold for the closing of *every* electron shell and in which the number 2 was considered to be as essential as 8 in contrast to Sommerfeld's approach.

Following Bohr's invitation, I went to Copenhagen in the autumn of 1922, where I made a serious effort to explain the so-called « anomalous Zeeman effect », as the spectroscopists called a type of splitting of the spectral lines in a magnetic field which is different from the normal triplet. On the one hand, the anomalous type of splitting exhibited beautiful and simple laws and Landé had already succeeded to find the simpler splitting of the spectroscopic terms from the observed splitting of the lines. The most fundamental of his results thereby was the use of half-integers as magnetic quantum numbers for the doublet-spectra of the alkali metals. On the other hand, the anomalous splitting was hardly understandable from the standpoint of the mechanical model of the atom, since very general assumptions concerning the electron, using classical theory as well as quantum theory, always led to the same triplet. A closer investigation of this problem left me with the feeling that it was even more unapproachable. We know now that at that time one was confronted with two logically different difficulties simultaneously. One was the absence of a general key to translate a given mechanical model into quantum theory which one tried in vain by using classical mechanics to describe the stationary quantum states themselves. The second difficulty was our ignorance concerning the proper classical model itself which could be suited to derive at all an anomalous splitting of spectral lines emitted by an atom in an external magnetic field. It is therefore not surprising that I could not find a satisfactory solution of the problem at that time. I succeeded, however, in generalizing Landé's term analysis for very strong magnetic fields[2], a case which, as a result of the magneto-optic transformation (Paschen-Back effect), is in many respects simpler. This early work was of decisive importance for the finding of the exclusion principle.

Very soon after my return to the University of Hamburg, in 1923, I gave there my inaugural lecture as *Privatdozent* on the Periodic System of Elements. The contents of this lecture appeared very unsatisfactory to me, since the problem of the closing of the electronic shells had been clarified no further. The only thing that was clear was that a closer relation of this problem to the theory of multiplet structure must exist. I therefore tried to examine again critically the simplest case, the doublet structure of the alkali spectra. According to the point of view then orthodox, which was also taken over by Bohr in his already mentioned lectures in Göttingen, a non-vanishing angular momentum of the atomic core was supposed to be the cause of this doublet structure.

In the autumn of 1924 I published some arguments against this point of view, which I definitely rejected as incorrect and proposed instead of it the assumption of a new quantum theoretic property of the electron, which I called a « two-valuedness not describable classically »[3]. At this time a paper of the English physicist, Stoner, appeared[4] which contained, besides improvements in the classification of electrons in subgroups, the following essential remark: For a given value of the principal quantum number is the number of energy levels of a single electron in the alkali metal spectra in an external magnetic field the same as the number of electrons in the closed shell of the rare gases which corresponds to this principal quantum number.

On the basis of my earlier results on the classification of spectral terms in a strong magnetic field the general formulation of the exclusion principle became clear to me. The fundamental idea can be stated in the following way: The complicated numbers of electrons in closed subgroups are reduced to the simple number *one* if the division of the groups by giving the values of the four quantum numbers of an electron is carried so far that every degeneracy is removed. An entirely non-degenerate energy level is already « closed », if it is occupied by a single electron; states in contradiction with this postulate have to be excluded. The exposition of this general formulation of the exclusion principle was made in Hamburg in the spring of 1925[5], after I was able to verify some additional conclusions concerning the anomalous Zeeman effect of more complicated atoms during a visit to Tübingen with the help of the spectroscopic material assembled there.

With the exception of experts on the classification of spectral terms, the physicists found it difficult to understand the exclusion principle, since no meaning in terms of a model was given to the fourth degree of freedom of the electron. The gap was filled by Uhlenbeck and Goudsmit's idea of electron spin[6], which made it possible to understand the anomalous Zeeman effect simply by assuming that the spin quantum number of one electron is equal to ½ and that the quotient of the magnetic moment to the mechanical angular moment has for the spin a value twice as large as for the ordinary orbit of the electron. Since that time, the exclusion principle has been closely connected with the idea of spin. Although at first I strongly doubted the correctness of this idea because of its classical-mechanical character, I was finally converted to it by Thomas' calculations[7] on the magnitude of doublet splitting. On the other hand, my earlier doubts as well as the cautious expression « classically non-describable two-valuedness » experienced a certain verification during later developments, since Bohr was able to show on the basis of wave mechanics that the electron spin cannot be measured by classically describable experiments (as, for instance, deflection of molecular beams in external electromagnetic fields) and must therefore be considered as an essentially quantum-mechanical property of the electron[8,9].

The subsequent developments were determined by the occurrence of the new quantum mechanics. In 1925, the same year in which I published my paper on the exclusion principle, De Broglie formulated his idea of matter waves and Heisenberg the new matrix-mechanics, after which in the next year Schrödinger's wave mechanics quickly followed. It is at present unnecessary to stress the importance and the fundamental character of these discoveries, all the more as these physicists have themselves explained, here in Stockholm, the meaning of their leading ideas[10]. Nor does time permit me to illustrate in detail the general epistemological significance of the new discipline of quantum mechanics, which has been done, among others, in a number of articles by Bohr, using hereby the idea of « complementarity » as a new central concept[11]. I shall only recall that the statements of quantum mechanics are dealing only with possibilities, not with actualities. They have the form « This is not possible » or « Either this or that is possible », but they can never say « That will actually happen then and there ». The actual observation appears as an event outside the range of a description by physical laws and brings forth in general a discontinuous selection out of the several possibilities foreseen by the statistical laws of the new theory. Only this renouncement concerning the old claims for an objective description of the physical phenomena, independent of the way in which they are observed, made it possible to reach again the self-consistency of quantum theory, which actually had been lost since Planck's discovery of the quantum of action. Without discussing further the change of the attitude of modern physics to such concepts as « causality » and « physical reality » in comparison with the older classical physics I shall discuss more particularly in the following the position of the exclusion principle on the new quantum mechanics.

As it was first shown by Heisenberg[12], wave mechanics leads to qualitatively different conclusions for particles of the same kind (for instance for electrons) than for particles of different kinds. As a consequence of the impossibility to distinguish one of several like particles from the other, the wave functions describing an ensemble of a given number of like particles in the configuration space are sharply separated into different classes of symmetry which can never be transformed into each other by external perturbations. In the term « configuration space » we are including here the spin degree of freedom, which is described in the wave function of a single particle by an index with only a finite number of possible values. For electrons this number is equal to two; the configuration space of N electrons has therefore 3 N space dimensions and N indices of « two-valuedness ». Among the different classes of symmetry, the most important ones (which moreover for two particles are the only ones) are the symmetrical class, in which the wave function does not change its value when the space and spin coordinates of two particles are permuted, and the antisymmetrical class, in which for such a permutation the wave function changes its sign. At this stage of the theory three different hypotheses turned out to be logically possible concerning the actual ensemble of several like particles in Nature.

I. This ensemble is a mixture of all symmetry classes.
II. Only the symmetrical class occurs.
III. Only the antisymmetrical class occurs.

As we shall see, the first assumption is never realized in Nature. Moreover, it is only the third assumption that is in accordance with the exclusion principle, since an antisymmetrical function containing two particles in the same state is identically zero. The assumption III can therefore be considered as the correct and general wave mechanical formulation of the exclusion principle. It is this possibility which actually holds for electrons.

This situation appeared to me as disappointing in an important respect. Already in my original paper I stressed the circumstance that I was unable to give a logical reason for the exclusion principle or to deduce it from more general assumptions. I had always the feeling and I still have it today, that this is a deficiency. Of course in the beginning I hoped that the new quantum mechanics, with the help of which it was possible to deduce so many half-empirical formal rules in use at that time, will also rigorously deduce the exclusion principle. Instead of it there was for electrons still an exclusion: not of particular states any longer, but of whole classes of states, namely the exclusion of all classes different from the antisymmetrical one. The impression that the shadow of some incompleteness fell here on the bright light of success of the new quantum mechanics seems to me unavoidable. We shall resume this problem when we discuss relativistic quantum mechanics but wish to give first an account of further results of the application of wave mechanics to systems of several like particles.

In the paper of Heisenberg, which we are discussing, he was also able to give a simple explanation of the existence of the two non-combining spectra of helium which I mentioned in the beginning of this lecture. Indeed, besides the rigorous separation of the wave functions into symmetry classes with respect to space-coordinates and spin indices together, there exists an approximate separation into symmetry classes with respect to space coordinates alone. The latter holds only so long as an interaction between the spin and the orbital motion of the electron can be neglected. In this way the para- and ortho-helium spectra could be interpreted as belonging to the class of symmetrical and antisymmetrical wave functions respectively in the space coordinates alone. It became clear that the energy difference between corresponding levels of the two classes has nothing to do with magnetic interactions but is of a new type of much larger order of magnitude, which one called exchange energy.

Of more fundamental significance is the connection of the symmetry classes with general problems of the statistical theory of heat. As is well known, this theory leads to the result that the entropy of a system is (apart from a constant factor) given by the logarithm of the number of quantum states of the whole system on a so-called energy shell. One might first expect that this number should be equal to the corresponding volume of the multidimensional phase space divided by h^f, where h is Planck's constant and f the number of degrees of freedom of the whole system. However, it turned out that for a system of N like particles, one had still to divide this quotient by $N!$ in order to get a value for the entropy in accordance with the usual postulate of homogeneity that the entropy has to be proportional to the mass for a given inner state of the substance. In this way a qualitative distinction between like and unlike particles was already preconceived in the general statistical mechanics, a distinction which Gibbs tried to express with his concepts of a generic and a specific phase. In the light of the result of wave mechanics concerning the symmetry classes, this division by $N!$, which had caused already much discussion, can easily be interpreted by accepting one of our assumptions II and III, according to both of which only one class of symmetry occurs in Nature. The density of quantum states of the whole system then really becomes smaller by a factor $N!$ in comparison with the density which had to be expected according to an assumption of the type I admitting all symmetry classes.

Even for an ideal gas, in which the interaction energy between molecules can be neglected, deviations from the ordinary equation of state have to be expected for the reason that only one class of symmetry is possible as soon as the mean De Broglie wavelength of a gas molecule becomes of an order of magnitude comparable with the average distance between two molecules, that is, for small temperatures and large densities. For the antisymmetrical class the statistical consequences have been derived by Fermi and Dirac[13], for the symmetrical class the same had been done already before the discovery of the new quantum mechanics by Einstein and Bose[14]. The former case could be applied to the electrons in a metal and could be used for the interpretation of magnetic and other properties of metals.

As soon as the symmetry classes for electrons were cleared, the question arose which are the symmetry classes for other particles. One example for particles with symmetrical wave functions only (assumption II) was already known long ago, namely the photons. This is not only an immediate consequence of Planck's derivation of the spectral distribution of the radiation energy in the thermodynamical equilibrium, but it is also necessary for the applicability of the classical field concepts to light waves in the limit where a large and not accurately fixed number of photons is present in a single quantum state. We note that the symmetrical class for photons occurs together with the integer value 1 for their spin, while the antisymmetrical class for the electron occurs together with the half-integer value ½ for the spin.

The important question of the symmetry classes for nuclei, however, had still to be investigated. Of course the symmetry class refers here also to the permutation of both the space coordinates and the spin indices of two like nuclei. The spin index can assume $2I + 1$ values if I is the spin-quantum number of the nucleus which can be either an integer or a half-integer. I may include the historical remark that already in 1924, before the electron spin was discovered, I proposed to use the assumption of a nuclear spin to interpret the hyperfine-structure of spectral lines[15]. This proposal met on the one hand strong opposition from many sides but influenced on the other hand Goudsmit and Uhlenbeck in their claim of an electron spin. It was only some years later that my attempt to interpret the hyperfine-structure could be definitely confirmed experimentally by investigations in which also Zeeman himself participated and which showed the existence of a magneto-optic transformation of the hyperfine-structure as I had predicted it. Since that time the hyperfine-structure of spectral lines became a general method of determining the nuclear spin.

In order to determine experimentally also the symmetry class of the nuclei, other methods were necessary. The most convenient, although not the only one, consists in the investigation of band spectra due to a molecule with two like atoms[16]. It could easily be derived that in the ground state of the electron configuration of such a molecule the states with even and odd values of the rotational quantum number are symmetric and antisymmetric respectively for a permutation of the space coordinates of the two nuclei. Further there exist among the $(2I + 1)^2$ spin states of the pair of nuclei, $(2I + 1)(I + 1)$ states symmetrical and $(2I + 1)I$ states antisymmetrical in the spins, since the $(2I+ 1)$ states with two spins in the same direction are necessarily symmetrical. Therefore the conclusion was reached: If the total wave function of space coordinates and spin indices of the nuclei is symmetrical, the ratio of the weight of states with an even rotational quantum number to the weight of states with an odd rotational quantum number is given by $(I+ 1) : I$. In the reverse case of an antisymmetrical total wave function of the nuclei, the same ratio is $I : (I + 1)$. Transitions between one state with an even and another state with an odd rotational quantum number will be extremely rare as they can only be caused by an interaction between the orbital motions and the spins of the nuclei. Therefore the ratio of the weights of the rotational states with different parity will give rise to two different systems of band spectra with different intensities, the lines of which are alternating.

The first application of this method was the result that the protons have the spin ½ and fulfill the exclusion principle just as the electrons. The initial difficulties to understand quantitatively the specific heat of hydrogen molecules at low temperatures were removed by Dennison's hypothesis[17], that at this low temperature the thermal equilibrium between the two modifications of the hydrogen molecule (ortho-H_2: odd rotational quantum numbers, parallel proton spins; para-H_2: even rotational quantum numbers, antiparallel spins) was not yet reached. As you know, this hypothesis was later, confirmed by the experiments of Bonhoeffer and Harteck and of Eucken, which showed the theoretically predicted slow transformation of one modification into the other.

Among the symmetry classes for other nuclei those with a different parity of their mass number M and their charge number Z are of a particular interest. If we consider a compound system consisting of numbers $A_1, A_2, \dots$ of different constituents, each of which is fulfilling the exclusion principle, and a number S of constituents with symmetrical states, one has to expect symmetrical or antisymmetrical states if the sum $A_1 + A_2 + \dots$ is even or odd. This holds regardless of the parity of S. Earlier one tried the assumption that nuclei consist of protons and electrons, so that M is the number of protons, $M - Z$ the number of electrons in the nucleus. It had to be expected then that the parity of Z determines the symmetry class of the whole nucleus. Already for some time the counter-example of nitrogen has been known to have the spin I and symmetrical states[18]. After the discovery of the neutron, the nuclei have been considered, however, as composed of protons and neutrons in such a way that a nucleus with mass number M and charge number Z should consist of Z protons and $M - Z$ neutrons. In case the neutrons would have symmetrical states, one should again expect that the parity of the charge number Z determines the symmetry class of the nuclei. If, however, the neutrons fulfill the exclusion principle, it has to be expected that the parity of M determines the symmetry class : For an even M, one should always have symmetrical states, for an odd M, antisymmetrical ones. It was the latter rule that was confirmed by experiment without exception, thus proving that the neutrons fulfill the exclusion principle.

The most important and most simple crucial example for a nucleus with a different parity of M and Z is the heavy hydrogen or deuteron with $M = 2$ and $Z = 1$ which has symmetrical states and the spin $I = 1$, as could be proved by the investigation of the band spectra of a molecule with two deuterons[19]. From the spin value I of the deuteron can be concluded that the neutron must have a half-integer spin. The simplest possible assumption that this spin of the neutron is equal to ½, just as the spin of the proton and of the electron, turned out to be correct.

There is hope, that further experiments with light nuclei, especially with protons, neutrons, and deuterons will give us further information about the nature of the forces between the constituents of the nuclei, which, at present, is not yet sufficiently clear. Already now we can say, however, that these interactions are fundamentally different from electromagnetic interactions. The comparison between neutron-proton scattering and proton-proton scattering even showed that the forces between these particles are in good approximation the same, that means independent of their electric charge. If one had only to take into account the magnitude of the interaction energy, one should therefore expect a stable di-proton or ${}^2_2\mathrm{He}$ ($M = 2, Z = 2$) with nearly the same binding energy as the deuteron. Such a state is, however, forbidden by the exclusion principle in accordance with experience, because this state would acquire a wave function symmetric with respect to the two protons. This is only the simplest example of the application of the exclusion principle to the structure of compound nuclei, for the understanding of which this principle is indispensable, because the constituents of these heavier nuclei, the protons and the neutrons, fullfil it.

In order to prepare for the discussion of more fundamental questions, we want to stress here a law of Nature which is generally valid, namely, the connection between spin and symmetry class. *A half-integer value of the spin quantum number is always connected with antisymmetrical states (exclusion principle), an integer spin with symmetrical states.* This law holds not only for protons and neutrons but also for protons and electrons. Moreover, it can easily be seen that it holds for compound systems, if it holds for all of its constituents. If we search for a theoretical explanation of this law, we must pass to the discussion of relativistic wave mechanics, since we saw that it can certainly not be explained by non-relativistic wave mechanics.

We first consider classical fields[20], which, like scalars, vectors, and tensors transform with respect to rotations in the ordinary space according to a one-valued representation of the rotation group. We may, in the following, call such fields briefly « one-valued » fields. So long as interactions of different kinds of field are not taken into account, we can assume that all field components will satisfy a second-order wave equation, permitting a superposition of plane waves as a general solution. Frequency and wave number of these plane waves are connected by a law which, in accordance with De Broglie's fundamental assumption, can be obtained from the relation between energy and momentum of a particle claimed in relativistic mechanics by division with the constant factor equal to Planck's constant divided by 2π. Therefore, there will appear in the classical field equations, in general, a new constant μ with the dimension of a reciprocal length, with which the rest-mass m in the particle picture is connected by $m = h\,\mu/c$, where c is the vacuum-velocity of light. From the assumed property of one-valuedness of the field it can be concluded, that the number of possible plane waves for a given frequency, wave number and direction of propagation, is for a non-vanishing μ always odd. Without going into details of the general definition of spin, we can consider this property of the polarization of plane waves as characteristic for fields which, as a result of their quantization, give rise to integer spin values.

The simplest cases of one-valued fields are the scalar field and a field consisting of a four-vector and an antisymmetric tensor like the potentials and field strengths in Maxwell's theory. While the scalar field is simply fulfilling the usual wave equation of the second order in which the term proportional to μ^2 has to be included, the other field has to fulfill equations due to Proca which are a generalization of Maxwell's equations which become in the particular case $\mu = 0$. It is satisfactory that for these simplest cases of one-valued fields the energy density is a positive definite quadratic form of the field-quantities and their first derivatives at a certain point. For the general case of one-valued fields it can at least be achieved that the total energy after integration over space is always positive.

The field components can be assumed to be either real or complex. For a complex field, in addition to energy and momentum of the field, a four-vector can be defined which satisfies the continuity equation and can be interpreted as the four-vector of the electric current. Its fourth component determines the electric charge density and can assume both positive and negative values. It is possible that the charged mesons observed in cosmic rays have integral spins and thus can be described by such a complex field. In the particular case of real fields this four-vector of current vanishes identically.

Especially in view of the properties of the radiation in the thermodynamical equilibrium in which specific properties of the field sources do not play any role, it seemed to be justified first to disregard in the formal process of field quantization the interaction of the field with the sources. Dealing with this problem, one tried indeed to apply the same mathematical method of passing from a classical system to a corresponding system governed by the laws of quantum mechanics which has been so successful in passing from classical point mechanics to wave mechanics. It should not be forgotten, however, that a field can only be observed with help of its interaction with test bodies which are themselves again sources of the field.

The result of the formal process of field quantization were partly very encouraging. The quantized wave fields can be characterized by a wave function which depends on an infinite sequence of (non-negative) integers as variables. As the total energy and the total momentum of the field and, in case of complex fields, also its total electric charge turn out to be linear functions of these numbers, they can be interpreted as the number of particles present in a specified state of a single particle. By using a sequence of configuration spaces with a different number of dimensions corresponding to the different possible values of the total number of particles present, it could easily be shown that this description of our system by a wave function depending on integers is equivalent to an ensemble of particles with wave functions symmetrical in their configuration spaces.

Moreover Bohr and Rosenfeld[21] proved in the case of the electromagnetic field that the uncertainty relations which result for the average values of the field strengths over finite space-time regions from the formal commutation rules of this theory have a direct physical meaning so long as the sources can be treated classically and their atomistic structure can be disregarded. We emphasize the following property of these commutation rules: All physical quantities in two world points, for which the four-vector of their joining straight line is spacelike commute with each other. This is indeed necessary for physical reasons because any disturbance by measurements in a world point P_1, can only reach such points P_2, for which the vector P_1P_2, is timelike, that is, for which $c\ (t_1 - t_2) > r_{12}$. The points P_2 with a spacelike vector P_1P_2 for which $c\ (t_1 - t_2) < r_{12}$ cannot be reached by this disturbance and measurements in P_1 and P_2 can then never influence each other.

This consequence made it possible to investigate the logical possibility of particles with integer spin which would obey the exclusion principle. Such particles could be described by a sequence of configuration spaces with different dimensions and wave functions antisymmetrical in the coordinates of these spaces or also by a wave function depending on integers again to be interpreted as the number of particles present in specified states which now can only assume the values 0 or 1. Wigner and Jordan[22] proved that also in this case operators can be defined which are functions of the ordinary space-time coordinates and which can be applied to such a wave function. These operators do not fulfil any longer commutation rules: instead of the difference, the sum of the two possible products of two operators, which are distinguished by the different order of its factors, is now fixed by the mathematical conditions the operators have to satisfy. The simple change of the sign in these conditions changes entirely the physical meaning of the formalism. In the case of the exclusion principle there can never exist a limiting case where such operators can be replaced by a classical field. Using this formalism of Wigner and Jordan I could prove under very general assumptions that a relativistic invariant theory describing systems of like particles with integer spin obeying the exclusion principle would always lead to the noncommutability of physical quantities joined by a spacelike vector[23]. This would violate a reasonable physical principle which holds good for particles with symmetrical states. In this way, by combination of the claims of relativistic invariance and the properties of field quantization, one step in the direction of an understanding of the connection of spin and symmetry class could be made.

The quantization of one-valued complex fields with a non-vanishing four-vector of the electric current gives the further result that particles both with positive and negative electric charge should exist and that they can be annihilated and generated in external electromagnetic field[22]. This pair-generation and annihilation claimed by the theory makes it necessary to distinguish clearly the concept of charge density and of particle density. The latter concept does not occur in a relativistic wave theory either for fields carrying an electric charge or for neutral fields. This is satisfactory since the use of the particle picture and the uncertainty relations (for instance by analyzing imaginative experiments of the type of the γ-ray microscope) gives also the result that a localization of the particle is only possible with limited accuracy[24]. This holds both for the particles with integer and with half-integer spins. In a state with a mean value E of its energy, described by a wave packet with a mean frequency $\nu = E/h$, a particle can only be localized with an error $\Delta x > hc/E$ or $\Delta x > c/\nu$. For photons, it follows that the limit for the localization is the wavelength; for a particle with a finite rest-mass m and a characteristic length $\mu^{-1} = \hbar/mc$, this limit is in the rest system of the center of the wave packet that describes the state of the particles given by $\Delta x > \hbar/mc$ or $\Delta x > \mu^{-1}$.

Until now I have mentioned only those results of the application of quantum mechanics to classical fields which are satisfactory. We saw that the statements of this theory about averages of field strength over finite space-time regions have a direct meaning while this is not so for the values of the field strength at a certain point. Unfortunately in the classical expression of the energy of the field there enter averages of the squares of the field strengths over such regions which cannot be expressed by the averages of the field strengths themselves. This has the consequence that the zero-point energy of the vacuum derived from the quantized field becomes infinite, a result which is directly connected with the fact that the system considered has an infinite number of degrees of freedom. It is clear that this zero-point energy has no physical reality, for instance it is not the source of a gravitational field. Formally it is easy to subtract constant infinite terms which are independent of the state considered and never change; nevertheless it seems to me that already this result is an indication that a fundamental change in the concepts underlying the present theory of quantized fields will be necessary.

In order to clarify certain aspects of relativistic quantum theory I have discussed here, different from the historical order of events, the one-valued fields first. Already earlier Dirac[25] had formulated his relativistic wave equations corresponding to material particles with spin ½ using a pair of so-called spinors with two components each. He applied these equations to the problem of one electron in an electromagnetic field. In spite of the great success of this theory in the quantitative explanation of the fine structure of the energy levels of the hydrogen atom and in the computation of the scattering cross section of one photon by a free electron, there was one consequence of this theory which was obviously in contradiction with experience. The energy of the electron can have, according to the theory, both positive and negative values, and, in external electromagnetic fields, transitions should occur from states with one sign of energy to states with the other sign. On the other hand there exists in this theory a four-vector satisfying the continuity equation with a fourth component corresponding to a density which is definitely positive.

It can be shown that there is a similar situation for all fields, which, like the spinors, transform for rotations in ordinary space according to two-valued representations, thus changing their sign for a full rotation. We shall call briefly such quantities « two-valued ». From the relativistic wave equations of such quantities one can always derive a four-vector bilinear in the field components which satisfies the continuity equation and for which the fourth component, at least after integration over the space, gives an essentially positive quantity. On the other hand, the expression for the total energy can have both the positive and the negative sign.

Is there any means to shift the minus sign from the energy back to the density of the four-vector? Then the latter could again be interpreted as charge density in contrast to particle density and the energy would become positive as it ought to be. You know that Dirac's answer was that this could actually be achieved by application of the exclusion principle. In his lecture delivered here in Stockholm[10] he himself explained his proposal of a new interpretation of his theory, according to which in the actual vacuum all the states of negative energy should be occupied and only deviations of this state of smallest energy, namely holes in the sea of these occupied states are assumed to be observable. It is the exclusion principle which guarantees the stability of the vacuum, in which all states of negative energy are occupied. Furthermore the holes have all properties of particles with positive energy and positive electric charge, which in external electromagnetic fields can be produced and annihilated in pairs. These predicted positrons, the exact mirror images of the electrons, have been actually discovered experimentally.

The new interpretation of the theory obviously abandons in principle the standpoint of the one-body problem and considers a many-body problem from the beginning. It cannot any longer be claimed that Dirac's relativistic wave equations are the only possible ones but if one wants to have relativistic field equations corresponding to particles, for which the value ½ of their spin is known, one has certainly to assume the Dirac equations. Although it is logically possible to quantize these equations like classical fields, which would give symmetrical states of a system consisting of many such particles, this would be in contradiction with the postulate that the energy of the system has actually to be positive. This postulate is fulfilled on the other hand if we apply the exclusion principle and Dirac's interpretation of the vacuum and the holes, which at the same time substitutes the physical concept of charge density with values of both signs for the mathematical fiction of a positive particle density. A similar conclusion holds for all relativsitic wave equations with two-valued quantities as field components. This is the other step (historically the earlier one) in the direction of an understanding of the connection between spin and symmetry class.

I can only shortly note that Dirac's new interpretation of empty and occupied states of negative energy can be formulated very elegantly with the help of the formalism of Jordan and Wigner mentioned before. The transition from the old to the new interpretation of the theory can indeed be carried through simply by interchanging the meaning of one of the operators with that of its hermitian conjugate if they are applied to states originally of negative energy. The infinite « zero charge » of the occupied states of negative energy is then formally analogous to the infinite zero-point energy of the quantized one-valued fields. The former has no physicial reality either and is not the source of an electromagnetic field.

In spite of the formal analogy between the quantization of the one-valued fields leading to ensembles of like particles with symmetrical states and to particles fulfilling the exclusion principle described by two-valued operator quantities, depending on space and time coordinates, there is of course the fundamental difference that for the latter there is no limiting case, where the mathematical operators can be treated like classical fields. On the other hand we can expect that the possibilities and the limitations for the applications of the concepts of space and time, which find their expression in the different concepts of charge density and particle density, will be the same for charged particles with integer and with half-integer spins.

The difficulties of the present theory become much worse, if the interaction of the electromagnetic field with matter is taken into consideration, since the well-known infinities regarding the energy of an electron in its own field, the so-called self-energy, then occur as a result of the application of the usual perturbation formalism to this problem. The root of this difficulty seems to be the circumstance that the formalism of field quantization has only a direct meaning so long as the sources of the field can be treated as continuously distributed, obeying the laws of classical physics, and so long as only averages of field quantities over finite space-time regions are used. The electrons themselves, however, are essentially non-classical field sources.

At the end of this lecture I may express my critical opinion, that a correct theory should neither lead to infinite zero-point energies nor to infinite zero charges, that it should not use mathematical tricks to subtract infinities or singularities, nor should it invent a « hypothetical world » which is only a mathematical fiction before it is able to formulate the correct interpretation of the actual world of physics.

From the point of view of logic, my report on « Exclusion principle and quantum mechanics » has no conclusion. I believe that it will only be possible to write the conclusion if a theory will be established which will determine the value of the fine-structure constant and will thus explain the atomistic structure of electricity, which is such an essential quality of all atomic sources of electric fields actually occurring in Nature.

1. A. Landé, *Z. Physik,* 5 (1921) 231 and *Z. Physik,* 7 (1921) 398, Physik. Z., 22 (1921) 417.
2. W. Pauli, *Z. Physik,* 16 (1923) 155.
3. W. Pauli, *Z. Physik,* 31 (1925) 373.
4. E. C. Stoner, *Phil. Mag.,* 48 (1924) 719.
5. W. Pauli, *Z. Physik,* 31 (1925) 765.
6. S. Goudsmit and G. Uhlenbeck, *Naturwiss.,* 13 (1925) 953, *Nature,* 117 (1926) 264.
7. L. H. Thomas, *Nature,* 117 (1926) 514, and *Phil. Mag.,* 3 (1927) 1. Compare also J. Frenkel, Z. *Physik,* 37 (1926) 243.
8. Compare *Rapport du Sixième Conseil Solvay de Physique, Paris, 1932,* pp. 217-225.
9. For this earlier stage of the history of the exclusion principle compare also the author's note in *Science,* 103 (1946) 213, which partly coincides with the first part of the present lecture.
10. The Nobel Lectures of W. Heisenberg, E. Schrödinger, and P. A. M. Dirac are collected in *Die moderne Atomtheorie,* Leipzig, 1934.
11. The articles of N. Bohr are collected in *Atomic Theory and the Description of Nature,* Cambridge University Press, 1934. See also his article « Light and Life », *Nature,* 131 (1933) 421, 457.
12. W. Heisenberg, *Z. Physik,* 38 (1926) 411 and 39 (1926) 499.
13. E. Fermi, *Z. Physik,* 36 (1926) 902.
    P. A. M. Dirac, *Proc. Roy. Soc. London,* A 112 (1926) 661.
14. S. N. Bose, *Z. Physik,* 26 (1924) 178 and 27 (1924) 384.
    A. Einstein, *Berl. Ber.,* (1924) 261; (1925) 1, 18.
15. W. Pauli, *Naturwiss.,* 12 (1924) 741.
16. W. Heisenberg, Z. *Physik,* 41 (1927) 239, F. Hund, Z. *Physik, 42* (1927) 39.
17. D. M. Dennison, *Proc. Roy. Soc. London,* A 115 (1927) 483.
18. R. de L. Kronig, *Naturwiss., 16* (1928) 335.
    W. Heitler und G. Herzberg, *Naturwiss.,* 17 (1929) 673.
19. G. N. Lewis and M. F. Ashley, *Phys. Rev.,* 43 (1933) 837.
    G. M. Murphy and H. Johnston, *Phys. Rev.,* 45 (1934) 550 and 46 (1934) 95.
20. Compare for the following the author's report in *Rev. Mod. Phys.,* 13 (1941) 203, in which older literature is given. See also W. Pauli and V. Weisskopf, *Helv. Phys. Acta,* 7 (1934) 809.
21. N. Bohr and L. Rosenfeld, *Kgl. Danske Videnskab. Selskab. Mat. Fys. Medd.,* 12 [8] (I933).
22. P. Jordan and E. Wigner, *Z. Physik,* 47 (1928) 63I.
    Compare also V. Fock, *Z. Physik,* 75 (1932) 622.
23. W. Pauli, *Ann. Inst. Poincaré,* 6 (1936) 137 and *Phys. Rev.,* 58 (1940) 716.
24. L. Landau and R. Peierls, *Z. Physik,* 69 (1931) 56.
    Compare also the author's article in *Handbuch der Physik,* 24, Part 1, 1933, Chap. A, § 2.
25. P. A. M. Dirac, *Proc. Roy. Soc. London,* A 117 (1928) 610.