Quantum Complementarity for the Superconducting Condensate and the Resulting Electrodynamic Duality.

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Abstract

Experiments with small capacitance Josephson junction arrays are described that demonstrate the quantum behavior of the phase of the superconducting condensate. This quantum behavior is based on the complementarity of phase and number for a condensate state of many bosons. A key factor to observation of this quantum behavior is the coupling of the superconducting condensate state to dissipation in the electrodynamic environment of the Josephson junction. It is shown how one-dimensional Josephson junction arrays can be used to design an environment with very weak coupling to dissipation, allowing for measurement of a condensate with well defined number. The well defined number is manifest in the Coulomb blockade of Cooper pair tunneling.

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1 Introduction

Bose-Einstein condensates, superfluids, lasers and superconductors are all examples in physics where a coherence arises when many bosons occupy the same quantum state – the condensate. Often we refer to this condensate as a “macroscopic wave function, or a “macroscopic quantum state”. However, in most cases the condensate has remarkably classical properties. In superconductors for example, we are able to measure the phase of the condensate (actually phase difference between two condensates, $\phi = \theta_2 - \theta_1$) with the Josephson effect. The Josephson effects tells us that a supercurrent, $I$ will flow between two weakly coupled superconductors with zero voltage drop. The supercurrent is proportional to the sine of the phase difference between the two superconducting condensates.

$$I = I_C \sin \phi \quad (1)$$

If there is a voltage drop over the tunnel junction, $V$, gauge invariance of the phase requires that,

$$V = \frac{\Phi_0}{2\pi} \frac{d\phi}{dt} \quad (2)$$

where $\Phi_0 = h/2e$ is the flux quantum. When we measure a supercurrent, we are making a measurement on a classical variable $\phi$. The electrodynamics of Josephson junctions in circuits with other elements such as resistors, capacitors, and inductors, is explained with classical (non-linear) electrodynamics based on the Josephson relations (1) and (2).

The quantum nature of the condensate becomes apparent when we consider the degree of freedom of the condensate that is complementary to the phase $\theta$, namely the particle number, $N$. Anderson [1] discussed the relationship between phase and number in the context of superconductors in the early years after the Josephson effect, citing the uncertainty relation.

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of phase and number, $\Delta \theta \Delta N \geq 1/2$. This is an uncertainty relation for macroscopic quantum physics. By macroscopic we mean quantum states containing many particles. However, the extent to which $\theta$ and $N$ are quantum variables is a controversial subject in the quantum optics community (see discussion in [2]). Indeed, it is perhaps not always so clear just how the mathematical quantities $N$ and $\theta$ represent measurable physical quantities.

With superconductors we are dealing with a condensate of charged bosons and for this reason there is a measurable quantity associated with number, which is electrostatic potential. Hence, the complementarity of phase and number lead to a complementarity of electric circuit variables flux, $\Phi$ and charge, $Q$. In small capacitance Josephson junction circuits, $\Delta \Phi \Delta Q \geq \hbar/2$. This uncertainty relation between measurable variables contains the constant of Nature $\hbar$, which "sets the scale" for quantum effects. In this sense the quantum mechanics of the Josephson junction is no more macroscopic than that of a single electron orbiting an atom. For the single electron the relevant physical variables, position and momentum, have the same uncertainty constant $\hbar$. However, the fact that we are examining the quantum behavior of collective degrees of freedom of the condensate, rather than the degrees of freedom of a single particle, means that what we typically consider to be macroscopic variables, like current (the flow of many electrons) and voltage (potential due to charge separation of many electrons) become quantum variables. This rather unusual situation allows us to explore quantum mechanics in new ways. With macroscopic quantum phenomena we have more freedom to engineer a quantum system, and perhaps, to apply the strange predictions of quantum theory in new and interesting ways.

One interesting aspect of this macroscopic quantum behavior of a Bose condensate is the ability to study how the process of measurement really effects the quantities we are trying to measure. When using quantum mechanics to describe nature, we can not avoid the question of measurement. Heisenberg’s own interpretation of the uncertainty principle was that it set fundamental limits on measurement. He expresses a rather extreme point of view when stating that “...one has to specify definite experiments with which one intends to measure 'the position of the electron' [ read 'the phase of the superconductor'], otherwise these words have no meaning [3].” Today we might say that when we interpret experiments with quantum mechanics, we use a semi-classical approach and this approach requires that the quantities which we measure are taken as classical variables. However, what determines if this or that variable is classical is not intrinsic to the quantum system we are studying, but rather to the way in which we measure it. Whether we measure phase or number of the condensate does not depend fundamentally on any property of the condensate itself, but rather on how we make the measurement. This point can be nicely demonstrated by showing how the influence of the external measurement circuitry, or the electrodynamic environment of the Josephson junction, determines whether we will measure a Josephson effect (classical, well defined phase) or a complementary effect known as the Coulomb blockade of Cooper pair tunneling.

2 Bloch Wave Description of a Josephson Junction

The Josephson junction consists of a thin insulating barrier separating two superconductors, where the overlapping area is large compared to the barrier separation. This geometry lends itself directly to a lumped-element description in terms of a parallel plate capacitance $C$, with a charge $Q = CV$, and associated energy $Q^2/2C$. The Josephson relations (1) and (2) give rise to a coupling energy, $U_J = -E_J \cos \phi$ associated with establishing a flow between the two condensates, $(dU_J/dt = IV)$. The Josephson coupling energy, $E_J = (\Phi_0/2\pi)IC$. With these two energies we can write down a Hamiltonian for the Josephson junction in terms of the flux, $\Phi = \int V dt$ and the charge, $Q$.

$$H = \frac{Q^2}{2C} - E_J \cos \left(2\pi \frac{\Phi}{\Phi_0}\right)$$

The quantum behavior arises when flux and the charge are treated as non-commuting variables in the
Hamiltonian above, $[\Phi, Q] = i\hbar$.

The quantum description of the single Josephson junction based on the Hamiltonian (3) above was first put forward by Averin, Zorin and Likharev, [4]. This Hamiltonian arises in many contexts as we have seen in this symposium. It describes the Cooper pair box, which is the basic unit of a charge qubit [5, 6]. It was also used to describe 1D lattice of Bose-Einstein condensates [7]. The Schrödinger equation with the Hamiltonian (3) is the Mathieu equation, and the wave functions are Bloch waves of the form $\psi(\Phi) = \exp iq\Phi/hu_s(\Phi)$. The Bloch wave vector, $q$ is called quasi-charge in the context of Josephson junctions, and the integer $s$ is a band index. The Bloch waves are a result of the fact that the energy depends on phase, which is a periodic variable. Analogies can be made between quasi-charge and crystal momentum [8, 9].

In order that we may observe any measurable consequences of this quantum description of the Josephson junction in terms of Bloch waves, it is necessary that we make the Josephson coupling energy, $E_J$ comparable to the charging energy, $E_C = e^2/2C$. When $E_J \gg E_C$, we find that the dependence of the energy eigenvalues on the quasi-charge is negligible, and the low-lying energy states, $u_s(\Phi)$ are to an excellent approximation the bound harmonic-oscillator states of a single minimum of the Josephson potential $U_J$, with energy $E_s = (s + 1/2)\hbar\omega_p$ where the “plasma” frequency is given by $\hbar\omega_p = \sqrt{SE_JE_C}$. Early experiments demonstrating quantum behavior of Josephson junctions proved the existence of these energy levels [10]. In the other extreme, $E_C \gg E_J$ we can understand the quantum mechanics of the Josephson junction in terms of a single particle-like picture, and it is enough to calculate tunneling rates to first order in $E_J$, which is the coupling term between the condensates differing by only one Cooper pair [11, 12]. The description in terms of Bloch waves becomes most useful when $E_J \sim E_C$.

The dependence of the ground state energy on the quasi-charge leads to interesting effects. The junction voltage can be considered as the derivative of the ground state energy with respect to quasi-charge (just as supercurrent is the derivative of the ground state energy with respect to flux). From this it follows that there exists a relation which is dual to the first Josephson relation (1).

$$V = \text{V}_C\text{saw}\chi$$  \hspace{1cm} (4)$$

where we introduce the dimensionless quasi-charge, $\chi = 2\pi q/2e$, and the saw function is a $2\pi$ periodic function which has an analytic closed form derivable from the properties of the Mathieu functions. Within this description, Averin, Zorin and Likharev made a bold proposition that one could bias the junction with a constant current in such a way that

$$I = \frac{2e}{2\pi} \frac{d\chi}{dt}$$ \hspace{1cm} (5)$$

. Under these bias conditions there would be oscillations of the junction voltage with frequency, $f$ related to the current in a fundamental way, $I = 2ef$.

The relations (4) and (5) bear a fascinating electrodynamic duality to the Josephson relations (1) and (2). This duality is intimately connected with the quantum mechanical complementarity of flux and charge. Not only is there interesting macroscopic quantum electrodynamics embedded in this simple model, but we also find that the supercurrent is not the only dramatic physical manifestation of quantum “coherence” of a condensate of charged bosons. Just as the phase, $\phi$ is a classical variable describing the supercurrent and associated dissipationless flow of electromagnetic energy through the condensate, so also can the dimensionless quasi-charge, $\chi$ become a classical variable with which one can describe energy flow through the condensate. In the case of classical behavior of $\chi$, however, the condensate has a rigidly fixed number, and quantum mechanics forbids measurement of the phase ($\Delta N \Delta \theta \geq 1/2$).

For our effective description to be valid, we need to make the temperature small compared to the superconducting energy gap, $k_B T \ll \Delta \sim 200\mu eV$ for Al. The single electron or quasi-particle degrees of freedom of the superconducting electrodes can be neglected at low temperatures due to the free energy difference between the odd and even parity of the condensate in the small electrode [13, 14]. One might also add that the temperature should be low such that $k_B T \ll \hbar\omega_p$. However this criteria is too rigid,
A “temperature” cannot be associated with our effective description of the Josephson junction because that description contains only two degrees of freedom (Q and Φ) and it is not reasonable to talk about the temperature of a system with only two degrees of freedom. The many other degrees of freedom which are important for this macroscopic quantum description of the Josephson junction are known as the “environment”, and the temperature of the environment is important. However, more important is how strongly the environmental degrees of freedom couple to the junction degrees of freedom.

3 Coupling to the Environment

The effect of the environment can be understood by the following simple arguments: Suppose we have a junction with $E_J \sim E_C$ at $T = 0$. With the current state of the art it is possible to make junctions quite reliably of Al, with $C \sim 0.5 \times 10^{-15}$ F. This corresponds to $E_C = 80\mu$eV, or $\omega_p = 3 \times 10^{11}$ sec$^{-1}$. When we connect leads to this junction as depicted in fig. 1a, we find that the capacitance of the junction is very small compared to the capacitance of the leads. In particular, on the time scale associated with the ground state energy, $1/\omega_p$, the potential at the junction is maintained over a length of lead approximately $2\pi c/\omega_p$ (the electromagnetic wave length). The capacitance of this length of lead will be approximately $C_{\text{lead}} = 2\pi \epsilon_0 \epsilon (2\pi c/\omega_p)$, which is the order of $10^{-12}$ F for the parameters given above. Connecting the leads to the junction thus adds a huge capacitance in parallel with the junction, effectively masking any effect due to the charging energy.

This simple “horizon” picture \[15, 16\] is an electrostatic argument. A more realistic electrodynamic model can take into account the distributed inductance and capacitance of the lead. Such a theory has been worked out for the case of an arbitrary linear electrodynamic environment connected to the junction \[17, 18, 19\]. Here the electromagnetic modes of the leads represent a bath of harmonic oscillators which are coupled to the quantum system using a formalism developed by Calderia and Leggett \[20\]. Within the context of this theory, quantum fluctuations of the phase result when the real part of the impedance as seen by the Josephson junction, $Z_e$ becomes larger than the quantum resistance, $R_Q = h/4e^2 = 6.45 k\Omega$. In this high impedance limit, the perturbation theory can calculate the Cooper pair tunneling rates only for the case of weak coupling, $E_J/E_C \ll \sqrt{R_Q/Z_e}$. For the interesting case, $E_J \sim E_C$ and $Z_e \gg R_Q$ the assumption in the theory, that of uncorrelated single Cooper pair tunneling, breaks down, and we are left with the current bias assertion of Averin, Zorin and Likharev, namely that the environment feeds a constant quasi-charge to the junction.

The interesting case, $E_J \sim E_C$ and $Z_e \gg R_Q$ is
that Nature prefers the Josephson effect.


However not easy to achieve with simple measurement leads for rather fundamental reasons. At the frequencies of interest, $\omega_p$ the leads can be modeled as a transmission line (see fig. 1b), with real impedance $Z_{\text{line}} = \sqrt{L_0/C_0} \sim Z_0/2\pi$, where $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\Omega$ is the free space impedance. It is a fact of nature that this radiation resistance is much less than the quantum resistance, because there ratio $Z_0/R_Q = 8\alpha$, where $\alpha = 1/137.03599$ is the fine structure constant. In short, radiation damping kills the quantum fluctuations of the phase and we find that Nature prefers the Josephson effect.

Nevertheless, achieving the interesting case $Z_e >> R_Q$ is not a hopeless task. With these quantum electronic circuits we have the possibility to engineer an environment with the correct impedance. Initial attempts to achieve a high impedance environment used small thin-film resistors close to the junction with $R \gg R_Q$ [21], and this technique did allow for observation of the Coulomb blockade of Cooper pair tunneling in a single Josephson junction [22]. However, the resistors heat up if the current becomes too large, and generally their dissipative nature causes problems. More recently we have used Josephson junction transmission lines to achieve the high impedance environment [23]. These transmission lines, depicted in fig. 1c, have the added advantage that their impedance can be tuned in situ, giving the experimentalist an extra knob for studying the influence of the environment.

The tuning of $E_J$ is possible because the Josephson junctions in the transmission line can be made as small DC SQUIDs (two identical junctions in parallel forming a loop with area $A$). When the loop is small enough, and if we consider small phase difference across the SQUID ($I < I_C$), the SQUID acts as a linear inductor with inductance $L_J = \Phi_0/2\pi I_C$, where $I_C$ depends on the external magnetic field, $B$ as $I_C(B) = I_{C0} [\cos(\pi BA/\Phi_0)]$. A circuit schematic of the single junction biased by the Josephson junction transmission line is shown in fig 1d. The Josephson transmission line characteristics are rather complex [24], but in the low frequency limit ($\omega < 1/\sqrt{L_J C_J}$) we find that the line impedance $Z_L = R_Q \sqrt{4EC/E_J \sqrt{C_J/C_0}}$ can become larger than $R_Q$ if we make the junctions closely spaced so that $C_J \gg C_0$. The impedance of the Josephson junction transmission line in this limit is not due to electromagnetic waves, but rather to Josephson plasmons, extending over several junctions. Hence, we can engineer an environment without using intrinsically dissipative elements, by substituting electromagnetic modes with Josephson plasmon modes.

4 Experimental Realization

Josephson junctions were made of Al with an AlO$_x$ tunnel barrier using the shadow evaporation technique. An electron microscope image of a typical sample is shown in fig. 2. The central junction is connected to the external measurement and biasing circuitry through 4 arrays of SQUIDs. The voltage across the central junction is measured with two of the arrays, and the bias is applied through the other two arrays. This 4 point measurement allows us to measure the current versus voltage ($I-V$) characteristic of the central junction, irrespective of the $I-V$ relation of the arrays. The central junction is not constructed in a SQUID geometry, and is thus not affected by an external magnetic field, $B$ applied perpendicular to the page. The external field effects only the SQUID junctions in the arrays. All junctions were fabricated in the same oxidation process. The time of oxidation was adjusted so that the central junction had Josephson coupling $E_J = 32\mu eV$ and the area of the central junction was designed so that the charging energy was approximately $E_C = 180\mu eV$ (assuming a specific capacitance $c_s = 45 \mu F/\mu m^2$), or $E_J/E_C \approx 0.2$. The SQUID junctions in the arrays had area roughly 6 times that of the central junction, resulting in 6 times the capacitance and 1/6 of the resistance. Thus the SQUID junctions had $E_J/E_C$ ratio roughly 36 times larger than the central junction. For the samples discussed below, the arrays had 65 junctions, and were 13 $\mu m$ long.

Figure 3 shows the $I-V$ curve of the central junction as the external magnetic field, $B$ is tuned. $B$ is measured in terms of the parameter $f = BA/\Phi_0$, which measures the fraction of a flux quantum that penetrates each SQUID loop. We see in fig. 3 that for
The distinct Coulomb blockade state measured in this single junction (curve $f = 0.49$ of fig. 3) indicates that we can achieve the situation where the coupled condensates are measured in such a way that $\chi$ becomes a classical variable, and the relations (4) and (5) describe the electrodynamics of the junction. These relations predict a static state ($d\chi/dt = 0$) with zero-current and voltage less than the critical voltage $V_C$. This “super-voltage” state is the dual to the supercurrent in typical Josephson junctions. At finite current ($I \propto d\chi/dt \neq 0$), the voltage across the junction should oscillate with frequency $\omega_B = 2\pi I/2e = \dot{\chi}$. These Bloch oscillations [4] are dual to the AC Josephson effect. The voltage oscillation frequency is far too fast to detect with our very low frequency, high impedance measurement setup. However, these high frequency oscillation will be rectified by the external circuit, giving a finite DC voltage. This rectification effect is dual to what occurs in ordinary Josephson junctions. In fact, the measured $I − V$ curve shown in fig. 3 (curve $f = 0.49$) has a remarkable dual-similarity to that predicted from the most simple model of a Josephson junction embedded in a dissipative external circuit.

5 The Resistively and Capacitively Shunted Josephson junction and its Dual

In the Resistively and Capacitively Shunted Junction (RCSJ) model [25][26], an ideal Josephson channel described by the relations (1) and (2) connected in parallel with a capacitance $C$ and a resistor $R$ as shown in fig. 4a. The sum of the current through each of the three parallel branches equals the external current. The current through the Josephson junction is given by equation (1). The current through the capacitor is $CdV/dt$, and the current through the resistor is $V/R$. We eliminate the voltage with equation (2) and arrive at a non-linear differential equation for the phase. The equation can be made dimensionless by taking the derivatives with respect to a dimensionless time, measured in units of a characteristic...
time \( \tau_J = \Phi_0/2\pi I_C R \). This simplifies the equation so that there is only one parameter characterizing the dynamical system.

\[
\beta_J \ddot{\phi} + \dot{\phi} + \sin \phi = \frac{I_{\text{ext}}}{I_C} \tag{6}
\]

Here the dot means differentiation with respect to dimensionless time, and the parameter \( \beta_J \) is given by,

\[
\beta_J = \frac{R^2 C}{\Phi_0/2\pi I_C} \tag{7}
\]

For a general value of \( \beta_J \) we simulate the time dependence of \( \phi \) numerically and determine the DC \( I - V \) curve by calculating the time average voltage \( \Phi_0 \left< \dot{\phi} \right> \) versus the parameter \( I_{\text{ext}} \). This model, with a frequency independent linear resistor \( R \), is the simplest way to include dissipation in the phase dynamics of the Josephson junction. The model works very well for real tunnel junctions shunted by thin-film resistors because the junction is very accurately described by a lumped element capacitance, and the shunt resistor is close to the junction so that it typically has negligible inductance.

\[
\beta_L \ddot{\chi} + \dot{\chi} + \text{saw} \chi = \frac{V_{\text{ext}}}{V_C} \tag{8}
\]

The parameter \( \beta_L \) is given by,

\[
\beta_L = \frac{G^2 L}{2e/2\pi V_C} \tag{9}
\]

In fact, for \( E_J \geq E_C \) the properties of the Mathieu functions are such that the saw function is well approximated by the sine function (\( \text{saw} \chi \sim \sin \chi \)) and the duality is complete.

Figure 3: The measured \( I - V \) curve of the single junction as the magnetic field is changed, tuning the impedance of the SQUID arrays.

Figure 4: The circuit schematic for a) the RCSJ model and b) the SRLJ model.

The dual to the RCSJ model is shown in fig. 4b. We call this model the Serially Resistive and Inductive Junction model (SRLJ) [27]. The parallel connection is replaced by a series connection, the capacitance is replaced by an inductance \( L \), the resistance is replaced by a conductance \( G = 1/R \) and the ideal Josephson channel is replaced by the ideal Coulomb blockade junction described by relations (4) and (5). The sum of the voltage drop across each element is equal to the external applied voltage. The voltage across the Coulomb blockade element is given by equation (4). The voltage across the inductor is \( LdI/dt \), and the voltage across the resistor is \( I/G \). We eliminate the current with equation (5) and arrive at a non-linear differential equation for the dimensionless quasi-charge \( \chi \). Again, the equation can be made dimensionless by taking the derivatives with respect to a dimensionless variable which is time measured in units of a characteristic time \( \tau_L = 2eR/2\pi V_C \). We end up with a dual equation,
The shape of the DC $I - V$ curve for these dual models is shown in fig. 5 for various values of the parameter $\beta$. The duality is reflected in that the current and voltage axes simply change roles as we go from one model to the other, and the $\beta$ parameter has a different meaning.

The measured $I - V$ curve ($f = 0.49$ of fig. 3) has remarkable similarity to the $I - V$ curve predicted from the simple SRLJ model (fig. 5, $\beta = 3$). In the experiment the voltage was measured across the junction only, and therefore we were able to trace out the region of negative differential resistance of the $I - V$ curve. The large inductance is actually realized by the Josephson junction arrays. The resistance actually involves all sorts of complicated dissipative processes involving Zener tunneling between Bloch energy bands [9, 28]. The SRLJ model is a limiting case of a model which takes into account the distributed nature of the one-dimensional Josephson junction array. Figure 7 shows the finite element extension of the SRLJ model to the distributed, one-dimensional case. Here, $C_0$ is

6 Josephson Junction Arrays in the Coulomb Blockade Limit

We have also made measurements on voltage biased Josephson junction SQUID arrays. For longer arrays we are able to observe an $I - V$ curve similar to that seen for the single junction even when $E_J \simeq E_C$. For the SQUID arrays, the observed threshold voltage is tuned by changing the ratio $E_J/E_C$, in qualitative agreement with the critical voltage predicted by the Bloch band picture of the single Josephson junction. Figure 6 shows the measured $I - V$ curve of an array with 255 junctions. The measurement was taken with a DC voltage bias, so we observe a jump from the zero current state to the finite current state, and retrapping to the zero current state at lower voltage. This hysteretic $I - V$ curve is qualitatively similar to that predicted by the SRLJ model for $\beta > 1$. 

Figure 5: Calculated $I - V$ curves from the RCSJ or SRLG model for various values of the damping parameter $\beta_J$ or $\beta_L$.

Figure 6: Measured $I - V$ curves of a voltage biased one-dimensional, small capacitance Josephson junction SQUID array at two different values of the magnetic field. The fit of the SRLJ model to the data gives the parameter $\beta_L$ from which we determine the magnitude of the inductive term.
the stray capacitance of each electrode to ground, and we are interested in the $x$ dependence of the quasi-charge, $\chi(x)$. In a manner similar to that for the SRLJ model, we can write a set of coupled first order difference equations that can be combined to one second order equation, which, when we take the continuum limit ($\Delta x \to 0$) becomes a sine-Gordon type equation [27, 29],

$$\partial_{zz} \chi + \partial_{t\tau} \chi + \alpha \partial_x \chi + \text{saw} \chi = 0 \quad (10)$$

Here the $\partial_{zz}$ means twice differentiation with respect to the dimensionless variable, $z = x / \lambda_L$ where $\lambda_L = \sqrt{2e / 2\pi v_c c_0}$, $v_c = V_{GC} / \Delta x$ is the critical electric field (critical voltage per unit length). The dimensionless time variable is $\tau = (v_0 / \lambda_S)t$, given in terms of the characteristic velocity $v_0 = 1 / \sqrt{c_0}$, where $l$ and $c_0$ are the inductance and capacitance per unit length. The damping parameter is given by $\alpha = r c_0 v_0 \lambda_S$.

With the replacement of $\sin \chi \simeq \text{saw} \chi$, equation (10) is the damped sine-Gordon equation. This equation and its derivation is exactly dual to the parallel Josephson junction array, where an identical equation for the Josephson phase results. The sine-Gordon equation admits soliton solutions, which describe well-known Josephson fluxons (vortices) in the Josephson case. In our case of the Coulomb blockade, we have topological objects called Cooper pair charge solitons. These kink solutions are $2\pi$ twists of the variable $\chi(x)$ over the length $\lambda_L$. The kinks describe the electrostatic potential ($\chi \ dx / dx$) and the $x$ component of the electric field ($\chi \ d^2 \chi / dx^2$) due to one excess charge quantum (Cooper pair) in the array. The dynamic model (10) is a transport model for electromagnetic energy flow through the condensate when the number is rigidly fixed, and the phase has large quantum fluctuations.

The boundary conditions for this model are given by the voltage bias, $V \partial_x \chi = V / V_{bh}$ applied at either end of the array, $x = 0$ and $x = S$, where the threshold voltage, $V_{bh} = 2 \lambda_L V_c$. In the Bloch wave description of the Josephson junction, the critical voltage is a function of the ratio $E_I / E_C$. In the limit $E_I \gg E_C$ this critical voltage goes exponentially to zero, and therefore, the soliton length, $\lambda_L \to \infty$. If the soliton length is larger than the array length, $\lambda_L \gg S$, the above model reduces to the simple SRLJ model. We have studied this soliton model for our arrays [27] and come to the conclusion that our arrays could well be in the limit $\lambda_L \gg S$. This conclusion is reached on the basis of two observations: The observed threshold voltage scales linearly with the array length, as predicted for this limit, and we do not observe any resonant features in the $I - V$ curves indicating solutions with multiple solitons in the array. Therefore, we have fit the $I - V$ curves of the voltage biased arrays in the Coulomb blockade regime using the SRLJ model.

The results of fitting to the SRLJ model are shown in fig. 6. We have used a slightly more complicated model of the resistance which more accurately describes the Zener tunneling threshold for the onset of dissipation [30, 27]. The resistance, $R$ is chosen to be tangent to the $I - V$ curve at twice the threshold voltage, and the parameter $\beta_S$ is adjusted to fit the observed amount of hysteresis. In this way we can extract the inductance in the SRLJ model. This fitting procedure is the appropriate way to extract the correct order of magnitude of the inductance $L$ with out making the model unduly complicated. Note that it is absolutely necessary that the inductive term exist if this dynamical model is to explain the observed hysteresis in the $I - V$ curve.

The fitted values of the inductance per cell are shown in fig. 8 for various values of the magnetic field. This inductance is far too large to be an electromagnetic inductance ($\sim 20 \ \mu H / \mu m$), but it can be explained by kinetic inductance arising from the mass (not charge) of the quasi-charge soliton. The kinetic energy of the charge soliton can be expressed in terms...

![Series connected SRLG elements with capacitance $C_0$ to ground. In the continuum limit $\Delta x \to 0$ the quasi-charge $\chi(x)$ obeys a sine-Gordon like equation.](image)
of the current as \( E_k = \frac{1}{2} m^* v^2 n_{CP} = \frac{1}{2} L_k I^2 \) where \( v \) and \( n_{CP} \) are the velocity and the average number of single excess Cooper pair in the array. The kinetic inductance is then

\[
L_k = \frac{m^* S^2}{e^2 n_{CP}} \quad (11)
\]

where \( m^* \) and \( e^* \) are the Cooper pair mass and charge, respectively. By assuming that the inductance found from fitting the SLRG model to the data is a kinetic inductance, we can use expression (11) to estimate the \( n_{CP} \). Figure 8 also shows a plot of the \( n_{CP} \) versus the the magnetic field. The \( n_{CP} \) is decreasing by two orders of magnitude as the \( f \) is changed from 0 \( \rightarrow \) 0.35. An \( n_{CP} < 1 \) implies that the single Cooper pair moves through and leaves the array before the next one is injected. This is consistent with the assumption of the long soliton limit \( S/\lambda_L \ll 1 \).

![Figure 8](image.png)

**Figure 8:** The inductance per cell found by fitting the SLRG model to the data (fig. 6). Interpreting this inductance as kinetic inductance gives an effective number of Cooper pairs in the array \( n_{CP} \) which carry the current. An \( n_{CP} < 1 \) reflects the fact that the single Cooper pair moves very fast through the array, exiting before the next single Cooper pair is injected.

The maximum velocity of the quasi-charge soliton can be estimated from \( v_0 = \frac{a}{\sqrt{L C_0}} \). Using the inductance from the fit and a ground capacitance per cell of \( C_0 = 9 \) aF the velocity is found to vary from 1–10 km/s. This velocity together with \( n_{CP} \) results in a maximum current \( I = 2 e v_0 n_{CP} / S \) varying from 250–0.1 pA as the frustration is increased, in agreement with the current scale measured in the experiment.

### 7 Conclusion

We have sketched the physical picture and shown some experimental evidence of an electrodynamic dual description of the superconducting condensate having it’s roots in the quantum mechanical complementarity of phase and number. Within this picture, one can describe transport through the condensate of charged bosons in terms of two semi-classical pictures which are complimentary to one another. In one extreme we have a the Josephson effect with the classical phase variable. In this paper we have described how the description is realized in the opposite extreme, where quasi-charge is a classical variable.

We are most familiar with measurements of the superconducting condensate where the phase is a classical variable. In this case we have the familiar properties of superconductors – flux quantization and the Josephson effects. The classical nature of the phase means that there are very large quantum fluctuations of the number. By large quantum fluctuations of the number we mean that there is no physical manifestation (or measurable effect) of the quantization of charge for the familiar measurements we make on superconductors.

There exists however another regime of measurement on the superconducting condensate, where the quasi-charge is a classical variable. In this regime we have the quantization of charge, and the Coulomb blockade effects. The classical nature of the quasi-charge means that there are very large quantum fluctuations of the phase, by which we mean that we can not measure the phase difference between the condensates. In this Coulomb blockade regime, there is no physical manifestation of condensate phase, such as flux quantization. Charge quantization is manifest in the Coulomb blockade regime, and it can be explained as a \( 2\pi \) twist of the quasi-charge. A non-linear dynamic equation for the quasi-charge can be derived which explains the measured transport properties in the Coulomb blockade regime.
There exists a rich electrodynamic duality between these two complementary regimes of measurement of the condensate. We summarize this duality by listing the dual quantities in Table 1 for the zero-dimensional and one dimensional cases discussed here.

References


[5] Per Delsing’s talk in this symposium (see K. Bladh et al. in this volume)

[6] Michel Devoret’s talk in this symposium (see D. Vion et al. in this volume)

[7] Bill Phillips talk in this symposium


### Phase description

- **Current and voltage relations**
  - \( I = I_C \sin \phi \)
  - \( V = (\Phi_0/2\pi) \partial_t \phi \)

- **Quasi-charge description**
  - \( V = V_C \sin \chi \)
  - \( I = (2e/2\pi) \partial_t \chi \)

#### Point junction

- **Damping parameter**: \( \beta_J = (2\pi/\Phi_0) I_C C/G^2 \)
- **Characteristic time**: \( \tau_J = \Phi_0/2 I_C R \)
- **Josephson inductance**: \( L_J = (\Phi_0/2\pi)/I_C \cos \phi \)
- **Coulomb capacitance**: \( C_C = (2e/2\pi)/V_C \cos \chi \)

#### Damping parameter:

- \( \beta_J = (2\pi/\Phi_0) I_C C/G^2 \)
- \( \beta_L = (2\pi/2e) V_C L/R^2 \)

#### Characteristic time:

- \( \tau_J = \Phi_0/2 I_C R \)
- \( \tau_L = 2\pi R/2\pi V_C \)

#### Josephson inductance:

- \( L_J = (\Phi_0/2\pi)/I_C \cos \phi \)

### Distributed junction

- **Damping parameter**: \( \alpha = 1/\sqrt{\beta_J} \)
- **Boundary conditions**: \( \partial_z \phi(\pm s/2, \tau) = \pm I/I_{th} \)
- **Soliton resonance**: \( V_n = \Phi_0 n v_0/S \)
- **Soliton injection threshold**: \( I_{th} = 2i_C \lambda_s \)
- **Coulomb capacitance**: \( C_C = (2e/2\pi)/V_C \cos \chi \)

### Table 1: A Comparison of some dual relations and parameters for the phase and quasi-charge descriptions.