Topological Defects and Phase Transitions
In two dimensions

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Situation of 2D systems in 1970’s

2D magnetic models:

\[ H/kT = K(1 - s(i).s(j)) \]
\[ s = (s_1...s_n), \quad |s| = 1 \]

Numerical simulations and high temperature series expansions indicated:

n=1: (Ising model): yes, transition (exact solution, L. Onsager, 1944)
n=2: (superfluid He films): maybe
n>2: probably not
n= infinity: NO phase transition (exact solution, H.E. Stanley, 1968)

This situation needed further study, especially for n=2 (Superfluid He\(^4\) film, etc.)
Figure 1, M Chester, L C Yang and J B Stephens, Phys Rev Lett 29, 211 (1972)
1D Ising model: Topology defined by $s_i = \pm 1$ and spin configurations by positions of these domain walls or "topological defects".

Planar rotor model:

$$s_i = |s|(\cos \theta_i, \sin \theta_i)$$

$$\Psi = s_x + is_y = |s|e^{i\theta_i} = |\Psi|e^{i\theta_i}.$$  

Invariant under

$$\theta_i \to \theta_i + 2\pi n_i, \quad n_i = 0, \pm 1, \pm 2, \ldots$$

Topology is a torus $\Rightarrow$ global (topological) excitations are "vortices".

$$\oint_C d\theta = 2\pi n$$

Energy and entropy of isolated vortex in system of size $L$:

$$\Delta E = \pi J \ln(L/a), \quad H/kT = (J/4kT)\Sigma(s_i-s_j)^2 = (K(T)/2)\Sigma(\theta_i-\theta_j)^2$$

$$\Delta S = k_B \ln(L^2/a^2).$$

Central quantity in statistical mechanics is the Free Energy, $F = E - TS$ because the probability of a configuration with free energy $F$ is

$$P \propto \exp(-\beta F)$$

$$P(vortex) \to \left(\frac{L}{a}\right)^{-(\pi K-2)} = \begin{cases} 0, & \pi K > 2, \\ 1, & \pi K < 2 \end{cases}$$

When $K(T)$ large, topological sector is stable and when $K(T)$ small, have transitions between topological sectors!
The Heisenberg model has $n = 3$ components, $\mathbf{s} = (\cos\theta, \sin\theta \cos\phi, \sin\theta \cos\phi)$ and there is one topological invariant $N = 0, \pm 1, \pm 2, \cdots$ where

$$N = \frac{1}{4\pi} \int d^2r \sin\theta(r) \left( \frac{\partial \theta(r)}{\partial x} \frac{\partial \phi(r)}{\partial y} - \frac{\partial \phi(r)}{\partial x} \frac{\partial \theta(r)}{\partial y} \right).$$

If we regard the direction of the magnetization in space as giving a mapping of the space on to the surface of a unit sphere, the invariant $N$ measures the number of times space encloses the unit sphere. This invariant is of no consequence in statistical mechanics because the energy barrier separating configurations with different values of $N$ is of order unity. Thus, there is no barrier between different topological sectors which implies that there is no ordered state for the 2D $n = 3$ Heisenberg magnet.

N.D. Mermin, H. Wagner, 1966
Vortex with $n=+1$. $n=-1$ vortex, fluid flow is in opposite direction.

$\Delta \theta = +2\pi$
Uniform superfluid velocity $u_s$ reduced by $h/mL$ when vortex goes across system.
RG flows in the (K, y) plane. The transition temperature $T_c(y)$ is the straight line on the right ending at $K=2/\pi$. 

\[
\frac{dK^{-1}}{dl} = 4\pi^3 y^2 + \mathcal{O}(y^4), \\
\frac{dy}{dl} = (2 - \pi K)y + \mathcal{O}(y^3).
\]
Crucial predictions of our theory

**Measured stiffness:**

\[ K^R(K_0, y_0) = K^R(K(l), y(l)), \]

\[ \xi_-(T) \sim \exp \left( b |t|^{-\frac{1}{2}} \right), \quad t < 0, \quad t \equiv \frac{T - T_c}{T_c} \]

**Correlation lengths:**

\[ \xi_+(T) \sim \exp \left( \frac{2\pi}{b} t^{-\frac{1}{3}} \right), \quad t > 0, \]

**Superfluid density:**

\[ \frac{\hbar^2 \rho_s^R(T)}{m^2 k_B T} = K^R(K_0, y_0) = K^R(K(\infty), y(\infty)) = K(\infty) = \frac{2}{\pi} + b \sqrt{\hbar}, \quad t < 0 \]

\[ \frac{\rho_s^R(T_c^-)}{T_c} = \frac{2m^2 k_B}{\pi \hbar^2} = 3.491 \times 10^{-8} \text{gm cm}^{-2} \text{ K}^{-1}. \]


Figure 2: DJ Bishop and JD Reppy, Phys Rev Lett 40, 1727 (1978)
Figure 3: DJ Bishop and JD Reppy, Phys Rev Lett 40, 1727 (1978).
Two possible orders in 2D crystal:

Translational order: Particle positions at $R = \mathbf{r} + \mathbf{u}(\mathbf{r})$

$r = ne_1 + me_2$ define ideal periodic lattice.

$\mathbf{u}(\mathbf{r})$ is displacement from ideal lattice position.

Order Parameters:

$$\rho_G(\mathbf{r}) = \exp(iG \cdot (\mathbf{r} + \mathbf{u}(\mathbf{r})))$$

Orientational order: Triangular lattice has 6 crystal axes $\pi/3$ apart.

Orientational order parameter:

$$\psi(\mathbf{r}) = \exp(6i\theta(\mathbf{r}))$$
Harmonic crystal in 2D described by elastic free energy

\[ \frac{H}{kT} = \frac{1}{2} \int d^2r (2\mu u_{ij}^2 + \lambda u_{kk}^2) \quad u_{ij}(\vec{r}) = \frac{1}{2} \left( \frac{\partial u_i(\vec{r})}{\partial r_j} + \frac{\partial u_j(\vec{r})}{\partial r_i} \right) \]

\[ C_G(\vec{r}) = \langle \rho_G(\vec{r})\rho_G^*(0) \rangle \sim r^{-\eta_G(T)} \]

\[ \rho_G(\vec{r}) = \exp(i\vec{G} \cdot (\vec{r} + \vec{u}(\vec{r}))) \]

\[ \eta_G(T) = \frac{k_BT G^2(3\mu + \lambda)}{4\pi\mu(2\mu + \lambda)} \]

Structure function:

\[ S(\vec{q}) = \langle \rho(\vec{q})\rho(-\vec{q}) \rangle = \sum_\vec{r} e^{i\vec{q} \cdot \vec{r}} \langle \exp[i\vec{q} \cdot (\vec{u}(\vec{r}) - \vec{u}(0))] \rangle \sim |\vec{q} - \vec{G}|^{-2 + \eta_G(T)} \]
Figure 2.19: D.R. Nelson ``Defects and Geometry in Condensed Matter Physics” (Cambridge University Press) 2002

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Gaussian theory says: (i) algebraic decay of translational order (Mermin-Wagner theorem)  
(ii) long range orientational order  
(iii) elastic moduli finite  

Need to identify excitations which will lead to isotropic liquid – dislocations & disclinations  

Dislocation:  
\[ \int d\vec{u} = \vec{b}(\vec{r}) = n(\vec{r})\vec{e}_1 + m(\vec{r})\vec{e}_2 \]  
\[ \vec{e}_1 = (1, 0) \]  
\[ \vec{e}_2 = (-1/2, \sqrt{3}/2) \]  

Disclination  
\[ \int d\theta = \frac{2\pi}{6} n \quad n = 0, \pm 1, \pm 2, \ldots \]  
\[ \vec{e}_3 = (-1/2, -\sqrt{3}/2) \]
Dislocation in square lattice. Burger’s vector $b$ amount 3x5 contour fails to close.
Reprinted with permission from Figure 1: K Zahn, R Lenke and G Maret, Phys Rev Lett 82, 2721 (1999). A plot of the correlation function $g_G(r) = r^{-\eta}$.
Orientational correlation function $g_6(r) = 1/r^{\eta_6(T)}$. Reprinted with permission from K Zahn, R Lenke and G Maret, Phys Rev Lett 82, 2721 (1999)

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Figure 1 reprinted with permission from Z Hadzibabic, P Kruger, M Cheneau, B Battelier and J Dalibard, Nature 441, 1118 (2006). Copyright 2006 Nature Publishing Group.
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