Allocation Games—the Deferred Acceptance Algorithm*

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My work is in a branch of mathematics called "game theory." Game theory is a mathematical study of conflict and cooperation between any number of rational decision-makers, or "players." As such, it is a very useful tool for economists, as a large part of their work involves situations with multiple players working for optimal solutions.

One type of problem involves "matching," that is, the allocation of items or partners, based upon their preferences.

In this example, we look at a set of boys and girls arranging their dates. First, each player ranks the members of the opposite sex in order of their desirability.

Boys' Preferences			
Adam	**Bob**	**Charlie**	**Don**
Mary	Jane	Mary	Mary
Jane	Mary	Kate	Kate
Kate	Kate	Jane	Jane

* My father, recognizing his limitations, chose to do his lecture as a simple lesson on the "Marriage Problem," that is, the Deferred Matching Algorithm, as he taught it for many years. —*Peter Shapley*

Girls' Preferences		
Mary	**Jane**	**Kate**
Adam	Adam	Don
Bob	Charlie	Charlie
Charlie	Don	Bob
Don	Bob	Adam

Then each boy approaches the first girl on their lists.

In the first round, Mary gets three offers, and holds on to Adam, while rejecting the other two. Jane receives one offer, from Bob, so she asks him to wait. Kate receives no offers, so at the end of the day she has nobody. The girls don't commit to anyone; they just say something like "hold on while I think about it."

	Round 1	Round 2	Round 3	Round 4	Round 5
Mary	**Adam** (Charlie & Don rejected)	**Adam** (no new proposal)	**Adam** (no new proposal)	**Adam** (Bob rejected)	**Adam** (no new proposal)
Kate	No proposal	**Don** (Charlie rejected)	**Don** (no new proposal)	**Don** (no new proposal)	**Don** (Bob rejected)
Jane	**Bob**	**Bob** (no new proposal)	**Charlie** (Bob rejected)	**Charlie** (no new proposal)	**Charlie** (no new proposal)

In the second round, the two boys who aren't being held onto approach the second girls on their lists. In this case, Kate receives both proposals, holds onto Don and sends Charlie away. Since nobody proposed to Mary or Jane, they hold onto their boys from the first round.

In the third round, Charlie (the only boy not currently held by a girl), asks Jane. Since Jane ranks Charlie ahead of Bob (who she's held since the first round), she releases Bob, and holds on to Charlie.

In the fourth round, Bob asks Mary, but is rejected by her since she ranks Adam higher. In the fifth round Bob asks his last choice, Kate, but she rejects him as well, since she ranks Don higher.

At this point, all three girls have boys on hold, and the one unattached boy has been rejected by all three girls. The process is done. Adam and Mary end up together, as do Don and Kate, and Charlie and Jane. Bob ends up alone.

Since no boy proposes to any girl after she has rejected him, this algorithm will reach a stable solution in a finite number of steps. In this case it took five rounds.

Since each attached boy has been rejected by all the girls who he ranked ahead of the one he ended up with, this match is stable, since no boy could improve his matching by switching to a partner he ranked higher than the one he ended up with.

This is a simple example, but with more players, it turns out that the resulting pairs can be different if the girls do the proposing rather than the boys. But both solutions are stable.

This works with any different numbers of boys and girls. It turns out that if the boys do the proposing, the resulting pairings would be better for the boys; while if the girls do the proposing, it ends up better overall for the girls.

This problem was stated as an idealized situation, but the algorithm I created, and extensions of it, have turned out to be useful in a number of real world problems. The mathematics involved in this example, besides devising the algorithm itself, is the proof that the algorithm works, and that it results in optimal, stable solutions. Applying this to real world situations has been the work of economists.

Portrait photo of professor Shapley by photographer Ulla Montan.