THE RENORMALIZATION GROUP AND CRITICAL PHENOMENA

Nobel lecture, 8 December 1982

by

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1. *Introduction*

This paper has three parts. The first part is a simplified presentation of the basic ideas of the renormalization group and the ε expansion applied to critical phenomena, following roughly a summary exposition given in 1972[1]. The second part is an account of the history (as I remember it) of work leading up to the papers in I971-1972 on the renormalization group. Finally, some of the developments since 197 1 will be summarized, and an assessment for the future given.

II. *Many Length Scales and the Renormalization Group*

There are a number of problems in science which have, as a common characteristic, that complex microscopic behavior underlies macroscopic effects.

In simple cases the microscopic fluctuations average out when larger scales are considered, and the averaged quantities satisfy classical continuum equations. Hydrodynamics is a standard example of this where atomic fluctuations average out and the classical hydrodynamic equations emerge. Unfortunately, there is a much more difficult class of problems where fluctuations persist out to macroscopic wavelengths, and fluctuations on all intermediate length scales are important too.

In this last category are the problems of fully developed turbulent fluid flow, critical phenomena, and elementary particle physics. The problem of magnetic impurities in non-magnetic metals (the Kondo problem) turns out also to be in this category.

In fully developed turbulence in the atmosphere, global air circulation becomes unstable, leading to eddies on a scale of thousands of miles. These eddies break down into smaller eddies, which in turn break down, until chaotic motions on all length scales down to millimeters have been excited. On the scale of millimeters, viscosity damps the turbulent fluctuations and no smaller scales are important until atomic scales are reached.[2]

In quantum field theory, "elementary" particles like electrons, photons, protons and neutrons turn out to have composite internal structure on all size scales down to 0. At least this is the prediction ofquantum field theory. It is hard to make observations of this small distance structure directly; instead the particle scattering cross sections that experimentalists measure must be interpreted

using quantum field theory. Without the internal structure that appears in the theory, the predictions of quantum field theory would disagree with the experimental findings.[3]

A critical point is a special example of' a phase transition. Consider, for example, the water-steam transition. Suppose the water and steam are placed under pressure, always at the boiling temperature. At the critical point: a pressure of'218 Atm and temperature of 374°C,[4] the distinction between water and steam disappears, and the whole boiling phenomenon vanishes. The principal distinction between water and steam is that they have different densities. As the pressure and temperature approach their critical values, the difference in density between water and steam goes to zero. At the critical point one finds bubbles of steam and drops of water intermixed at all size scales from macroscopic, visible sizes down to atomic scales. Away from the critical point, surface tension makes small drops or bubbles unstable; but as water and steam become indistinguishable at the critical point, the surface tension between the two phases vanishes. In particular, drops and bubbles near micron sizes cause strong light scattering, called "critical opalescence", and the water and steam become milky.

In the Kondo effect, electrons of all wavelengths from atomic wavelengths up to very much larger scales, all in the conduction band of a metal, interact with the magnetic moment of each impurity in the metal.[5]

Theorists have difficulties with these problems because they involve very many coupled degrees of freedom. It takes many variables to characterize a turbulent flow or the state of a fluid near the critical point. Analytic methods are most effective when functions of only one variable one degree of freedom) are involved. Some extremely clever transformations have enabled special cases of the problems mentioned above to be rewritten in terms of independent degrees of freedom which could be solved analytically. These special examples include Onsager's solution of the two dimensional Ising model of a critical point, [6]the solution of Andrei and Wiegmann of the Kondo problem,[7] the solution of' the Thirring model of a quantum field theory,[8] and the simple solutions of noninteracting quantum fields. 'These are however only special cases; the entire problem of' fully developed turbulence, many problems in critical phenomena and virtually all examples of strongly coupled quantum fields have defeated analytic techniques up till now.

Computers can extend the capabilities of' theorists, but even numerical computer methods are limited in the number of degrees of freedom that are practical. Normal methods of' numerical integration fail beyond only 5 to 10 integration variables; partial differential equations likewise become extremely difficult beyond 3 or so independent variables. Monte Carlo and statistical averaging methods can treat some cases of' thousands or even millions of variables but the slow convergence of these methods versus computing time used is a perpetual hassle. An atmospheric flow simulation covering all length scales of turbulence would require a grid with millimeter spacing covering thousands of miles horizontally and tens of miles vertically: the total number of grid points would be of order 10^{25} far beyond the capabilities of any present or conceivable computer.

The "renormalization group" approach is a strategy for dealing with problems involving many length scales. The strategy is to tackle the problem in steps, one step for each length scale. In the case ofcritical phenomena, the problem, technically, is to carry out statistical averages over thermal fluctuations on all size scales. The renormalization group approach is to integrate out the fluctuations in sequence starting with fluctuations on an atomic scale and then moving to successively larger scales until fluctuations on all scales have been averaged out,

To illustrate the renormalization group ideas the case of' critical phenomena will be discussed in more detail. First the mean field theory of Landau will be described, and important questions defined. The renormalization group will be presented as an improvment to Landau's theory.

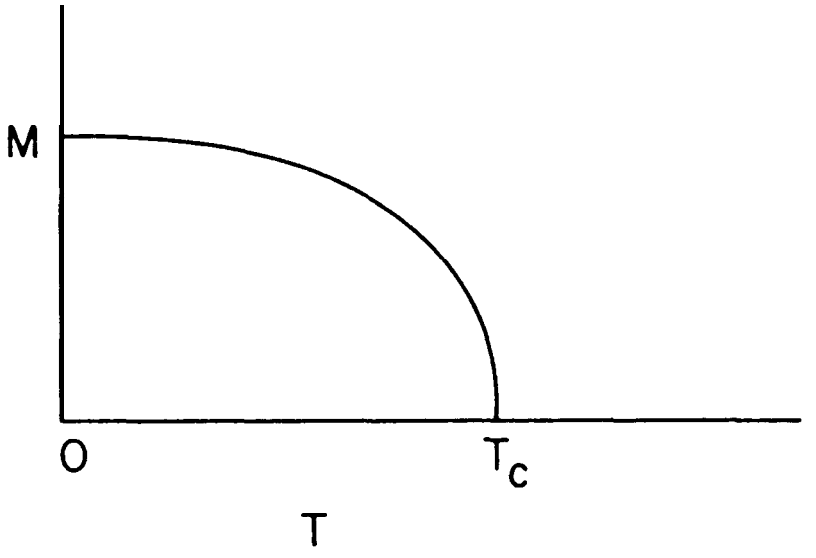
The Curie point of a ferromagnet will be used as a specific example of a critical point. Below the Curie temperature, an ideal ferromagnet exhibits spontaneous magnetization in the absence of' an external magnetic field; the direction of' the magnetization depends on the history of the magnet. Above the Curie temperature T_C, there is no spontaneous magnetization. Figure 1 shows a typical plot of' the spontaneous magnetization versus temperature. Just below the Curie temperature the magnetization is observed to behave as $(T_C - T)^{\beta}$, where β is an exponent somewhere near l/3 (in three dimensions).[9.10]

Magnetism is caused at the atomic level by unpaired electrons with magnetic moments, and in a ferromagnet, a pair of nearby electrons with moments aligned has a lower energy than if the moments are anti-aligned.[10] At high temperatures, thermal fluctuations prevent magnetic order. As the temperature is reduced towards the Curie temperature, alignment of one moment causes preferential alignment out to a considerable distance called the correlation length ξ. At the Curie temperature, the correlation length becomes infinite, marking the onset of preferential alignment of the entire system. Just above T_c, the correlation length is found to behave as $(T-T_c)^{-\nu}$, where ν is about 2/3 (in three dimensions).[11]

A simple statistical mechanical model of a ferromagnet involves a Hamiltonian which is a sum over nearest neighbor moment pairs with different energies for the aligned and antialigned case. In the simplest case, the moments are allowed only to be positive or negative along a fixed spatial axis; the resulting model is called the Ising model.[12]

The formal prescription for determining the properties of' this model is to compute the partition function Z, which is the sum of the Boltzmann factor exp(-H/kT) over all configurations of the magnetic moments, where k is Boltzmann's constant. The free energy F is proportional to the negative logarithm of Z.

The Boltzmann factor exp(-H/kT) is an analytic function of T near T_c, in fact for all T except T = 0. A sum of analytic functions is also analytic. Thus it is puzzling that magnets (including the Ising model) show complex non-analytic behavior at $T = T_c$. The true non-analytic behavior occurs only in the thermodynamic limit of a ferromagnet ofinfinite size; in this limit there are an infinite number of configurations and there are no analyticity theorems for the infinite sums appearing in this limit. However, it is difficult to understand how even an infinite sum can give highly non-analytic behavior. A major challenge has been to show how the non-analyticity develops.



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Landau's proposal[13] was that if only configurations with a given magnetization density M are considered then the free energy is analytic in M. For small M, the form of the free energy (to fourth order in M) is (from the analyticity assumption)

$$F = V\{RM^2 + UM^4\} \tag{1}$$

where V is the volume of the magnet and R and U are temperature-dependent constants. (A constant term independent of M has been omitted). In the absence of an external magnetic field, the free energy cannot depend on the sign of M, hence only even powers of M occur. The true free energy is the minimum of F over all possible values of M. In Landau's theory, R is 0 at the critical temperature, and U must be positive so that the minimum of F occurs at M = 0 when at the critical temperature. The minimum of F continues to be at M = 0 if R is positive: this corresponds to temperatures above critical. If R is negative the minimum occurs for non-zero M, namely the M value satisfying

$$0 = \frac{\partial F}{\partial M} = (2RM + 4UM^3) \tag{2}$$

or

$$M = \sqrt{-R/(2U)} \tag{3}$$

This corresponds to temperatures below critical.

Along with the analyticity of the free energy in M, Landau assumed analyticity in T, namely that R and U are analytic functions of T. Near T_c this means that

to a first approximation, U is a constant and R (which vanishes at T_c) is proportional to $T-T_c$ (It is assumed that dR/dT does not vanish at T_c). Then, below T_c, the magnetization behaves as

$$M \propto (T_c - T)^{1/2} \tag{4}$$

i.e. the exponent β is 1/2 which disagrees with the evidence, experimental and theoretical, that β is about 1/3.[9]

Landau's theory allows for a slowly varying space-dependent magnetization. The free energy for this case takes the Landau-Ginzburg form[14]

$$F = \int d^3x \{[\nabla M(x)]^2 + RM^2(x) + UM^4(x) - B(x)M(x)\} \tag{5}$$

where B(x) is the external magnetic field. The gradient term is the leading term in an expansion involving arbitrarily many gradients as well as arbitrarily high powers of M. For slowly varying fields M(x) higher powers of gradients are small and are neglected. (Normally the $\nabla M^2(x)$ term has a constant coefficient - in this paper this coefficient is arbitrarily set to 1) . One use of this generalized free energy is to compute the correlation length ξ above T_c. For this purpose let B(x) be very small δ function localized at x = 0. The U term in F can be neglected, and the magnetization which minimize the free energy satisfies

$$-\nabla^2 M(x) + RM(x) = B\delta^3(x) \tag{6}$$

The solution M(x) is

$$M(x) \propto Be^{-\sqrt{R}|\vec{x}|}/|\vec{x}| \tag{7}$$

and the correlation length can be read off to be

$$\xi \propto 1/\sqrt{R} \tag{8}$$

Hence near T_c, ξ is predicted to behave as $(T-T_c)^{-1/2}$, which again disagrees with experimental and theoretical evidence.

The Landau theory assumes implicitly that analyticity is maintained as all space-dependent fluctuations are averaged out. The loss of analyticity arises only when averaging over the values of the overall average magnetization M. It is this overall averaging, over $e^{-F/kT}$, which leads to the rule that F must be minimized over M, and the subsequent non-analytic formula (4) for M. To be precise, if the volume of the magnet is finite, $e^{-F/kT}$ must be integrated over M, with analytic results. It is only in the thermodynamic limit $V \to \infty$ that the average of $e^{-F/kT}$ is constructed by minimizing F with respect to M. and the nonanalyticity of Eqn. (4) occurs.

The Landau theory has the same physical motivation as hydrodynamics. Landau assumes that only fluctuations on an atomic scale matter. Once these have been averaged out the magnetization M(x) becomes a continuum, continous function which fluctuates only in response to external space-dependent stimuli. M(x) (or, if it is a constant, M) is then determined by a simple classical equation. Near the critical point the correlation function is itself the solution of the classical equation (6).

In a world with greater than four dimensions, the Landau picture is correct.[15] Four dimensions is the dividing line - below four dimensions, fluctuations on all scales up to the correlation length are important and Landau theory breaks down, [16]as will be shown below. An earlier criterion by Ginzburg[15] also would predict that four dimensions is the dividing line.

The role of long wavelength fluctuations is very much easier to work out near four dimensions where their effects are small. This is the only case that will be discussed here. Only the effects ofwavelengths long compared to atomic scales will be discussed, and it will be assumed that only modest corrections to the Landau theory are required. For a more careful discussion see ref. 17.

Once the atomic scale fluctuations have been averaged out, the magnetization is a function M(x) on a continuum, as in Landau theory. However, long wavelength fluctuations are still present in M(x) - they have not been averaged out - and the allowed forms of 14(x) must be stated with care. To be precise, suppose fluctuations with wavelengths < $2\pi L$ have been averaged out, where L is a length somewhat larger than atomic dimensions. Then M(x) can contain only Fourier modes with wavelengths > $2\pi L$. This requirement written out, means

$$M(\vec{x}) = \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} M_{\vec{k}} \tag{9}$$

where the integral over $\vec{k}$ means $(2\pi)^{-d}\int d^d k$, d is the number of space dimensions, and the limit on wavelengths means that the integration over k is restricted to values of $\vec{k}$ with $|\vec{k}| < L^{-1}$.

Averaging over long wavelength fluctuations now reduces to integrating over the variables $M_{\vec{k}}$, for all $|\vec{k}| < L^{-1}$. There are many such variables; normally this would lead to many coupled integrals to carry out, a hopeless task. Considerable simplifications will be made below in order to carry out these integrations.

We need an integrand for these integrations. The integrand is a constrained sum of the Boltzmann factor over all atomic configurations. The constraints an that all $M_{\vec{k}}$ for $|\vec{k}| < L^{-1}$ are held fixed. This is a generalization of the constrained sum in the Landau theory; the difference is that in the Landau theory only the average magnetization is held fixed. The result of the constrained sum will be written e^{-F}, similarly to Landau theory except for convenience the exponent is written F rather than F/kT (i.e. the factor 1 /kT is absorbed into an unconventional definition of F). The exponent F depends on the magnetization function M(x) of Eq. (9). We shall assume Landau's analysis is still valid for the form of F, namely F is given by Eq. (5). However, the importance of long wavelength fluctations means that the parameters R and U depend on L. Thus F should be denoted F_L:

$$F_L = \int d^d x \{(\nabla M)^2(x) + R_L M^2(x) + U_L M^4(x)\} \tag{10}$$

(in the absence of any external field) (in the simplified analysis presented here, the coefficient of $\nabla M^2(x)$ is unchanged at 1). The assumption will be reviewed later.

The L dependence of R_L and U_L will be determined shortly. However, the breakdown of analyticity at the critical point is a simple consequence of this L dependence. The L dependence persists only out to the correlation length ξ: fluctuations with wavelengths $>\xi$ will be seen to be always negligible. Once all wavelengths of fluctuations out to $L \sim \xi$ have been integrated out, one can use the Landau theory; this means (roughly speaking) substituting R_ξ and U_ξ in the formulae (4) and (8) for the spontaneous magnetization and the correlation length. Since ξ is itself non-analytic in T at T = T, the dependence of R_ξ and U_ξ on ξ introduces new complexities at the critical point. Details will be discussed shortly.

In order to study the effects of fluctuations, only a single wavelength scale will be considered; this is the basic step in the renormalization group method. To be precise, consider only fluctuations with wavelengths lying in an infinitesimal interval L to $L+\delta L$. To average over these wavelengths of fluctuations one starts with the Boltzmann factor e^{-F_L} where the wavelengths between L and $L+\delta L$ are still present in M(x), and then averages over fluctuations in M(x) with wavelengths between L and $L+\delta L$. The result of these fluctuation averages is a free energy F_L+bl. for a magnetization function (which will be denoted $M_H(x)$) with wavelengths > $L+\delta L$ only. The Fourier components of $M_H(x)$ are the same $\vec{k}$ that appear in M(x) except that $|\vec{k}|$ is now restricted to be less than $1/(L+\delta L)$.

The next step is to count the number of integration variables $M_{\vec{k}}$ with $|\vec{k}|$ lying between $1/L$ and $1/(L+\delta L)$. To make this count it is necessary to consider a finite system in a volume V. Then the number of degrees of freedom with wavelengths between $2\pi L$ and $2\pi(L+\delta L)$ is given by the corresponding phase space volume, namely the product of k space and position space volumes. This product is (apart from constant factors like π, etc.) $L^{-(d+1)}V\delta L$.

It is convenient to choose the integration variables not to be the $M_{\vec{k}}$ themselves but linear combinations which correspond to localized wave packets instead of plane waves. That is, the difference $M_H(x)-M(x)$ should be expanded in a set of wave packet functions $\psi_n(x)$, each of which has momenta only in the range $1/L$ to $1/(L+\delta L)$, but which is localized in x space as much as possible. Since each function $\psi_n(x)$ must (by the uncertainty principle) fill unit volume in phase space, the position space volume for each $\psi_n(x)$ is

$$\delta V = L^{d+1}/\delta L \tag{11}$$

and there are $V/\delta V$ wavefunctions $\psi_n(x)$. We can write

$$M(x) = M_H(x)+\Sigma m_n \psi_n(x) \tag{12}$$

and the integrations to be performed are integrations over the coefficients m_n.

Because of the local nature of the Landau-Ginzburg free energy, it will be assumed that the overlap of the different wavefunctions ψ_n can be neglected. Then each m_n integration can be treated separately, and only a single such integration will be discussed here. For this single integration, the form of M(x) can be written

$$M(x) = M_H(x)+m\psi(x) \tag{13}$$

since only one term from the sum over n contributes within the spatial volume occupied by the wavefunction $\psi(x)$.

The other simplification that will be made is to treat $M_H(x)$ as if it were a constant over the volume occupied by $\psi(x)$. In other words the very long wavelengths in $M_H(x)$ are emphasized relative to wavelengths close to L.

The calculation to be performed is to compute

$$e^{-F_{L+\delta L}[M_H]} = \int_I^\infty dm\, e^{-F_L[M_H + m\psi]} \tag{14}$$

where $F_{L+\delta L}$ and F_L, involve integration only over the volume occupied by $\psi(x)$. In expanding out $F_L[M_H+m\psi]$ the following simplifications will be made. First, all terms linear in $\psi(x)$ are presumed to integrate to 0 in the x integration defining F_L. Terms of third order and higher in ψ are also neglected. The function $\psi(x)$ is presumed to be normalized so that

$$\int d^d x\, \psi^2(x) = 1 \tag{15}$$

and due to the limited range ofwavelengths in $\psi(x)$, there results

$$\int [\nabla\psi(x)]^2 d^d x \simeq 1/L^2 \tag{16}$$

The result of these simplifications is that the integral becomes

$$e^{-F_{L+\delta L}[M_H]} = e^{-F_L[M_H]} \int_{-\infty}^{\infty} dm\, \exp\{(R_L + \tfrac{1}{L^2})\, m^2 + 6U_L M_H^2 m^2\} \tag{17}$$

or

$$F_{L+\delta L}[M_H] = F_L\{M_H\} + \frac{1}{2}\ell n\left(\frac{1}{L^2} + R_L + 6U_L M_H^2\right) \tag{18}$$

The logarithm must be rewritten as an integral over the volume occupied by $\psi(x)$; this integral can then be extended to an integral over the entire volume V when the contributions from all other m_n integrations are included. Also the logarithm must be expanded in powers of M_H; only the M_H^2 and M_H^4 terms will be kept. Further it will be assumed that R_L, changes slowly with L. When L is at the correlation length $\xi, 1/L^2$ and R_L, are equal (as already argued) so that for values of L intermediate between atomic sizes and the correlation length, R_L, is small compared to $1/L^2$. Expanding the logarithm in powers of R_L+GUt,Mi, to second order (to obtain an M_H^4 term) gives (of. Eq. (11)):

$$\begin{aligned}\frac{1}{2}\ell n\left(\frac{1}{L^2} + R_L + 6U_L M_H^2\right) &= \text{terms independent of } M_H \\ &+ (\delta V)(\delta L) L^{-d-1}\{3U_L M_H^2 L^2 - 9U_L^2 M_H^4 L^4 - 3R_L U_L M_H^2 L^4\}\end{aligned} \tag{19}$$

One can rewrite δV as an integral over the volume δV. There results the equations

$$R_{L+\delta L} = R_L + (3U_L L^{1-d} - 3R_L U_L L^{3-d})\delta L \tag{20}$$

$$U_{L+\delta L} = U_L - 9U_L^2 L^{3-d}\delta L \tag{21}$$

or

$$L\frac{dR_L}{dL} = 3L^{2-d}U_L - 3R_L U_L \cdot L^{4-d} \tag{22}$$

$$L\frac{dU_L}{dL} = -9U_L^2 L^{4-d} \tag{23}$$

These equations are valid only for $L < \xi$; for $L > \xi$ there is very little further change in R_L, or U_L, due to the switchover in the logarithm caused by the dominance of R_L rather than $1/L^2$. If d is greater than 4, it can be seen that R_L and U_L are constant for large L, as expected in the Landau theory. For example, if one assumes R_L. and U_L. are constant for large L it is easily seen that integration of (22) and (23) only gives negative powers of L. For d<4 the solutions are not constant. Instead, U_L behaves for sufficiently large L as

$$U_{L.} \simeq \frac{(4-d)}{9} L^{d-4} \tag{24}$$

(which is easily seen to be a solution of (23)), R_L. satisfies the equation

$$\frac{dR_L}{dL} + \frac{(4-d)}{3L} R_L = \frac{(4-d)}{3} L^{-3} \tag{25}$$

whose solution is

$$R_L = cL^{(d-4)/3} - \frac{(4-d)}{3} \frac{1}{2-(4-d)/3} L^{-2} \tag{26}$$

where c is related to the value of R_L. at some initial value of L. For large enough L, the L^{-2} term can be neglected.

The parameter c should be analytic in temperature, in fact proportional to T-T,. Hence, for large L

$$R_{L.} \propto L^{(d-4)/3}(T-T_c) \tag{27}$$

which is analytic in T for fixed L. However the equation for ξ is

$$\xi \propto R_\xi^{-1/2} = (T-T_c)^{-1/2} \xi^{(4-d)/6} \tag{28}$$

Let

$$\varepsilon = 4\text{-}d \tag{29}$$

then the correlation length exponent is

$$\upsilon = \frac{1}{2}\frac{1}{1-\varepsilon/6} \tag{30}$$

which gives $\upsilon = 0.6$ in 3 dimensions. Similarly, the spontaneous magnetization below T_c behaves as $(R_\xi/U_\xi)^{1/2}$ giving

$$\beta = \frac{1}{2} - \frac{\varepsilon}{3}\frac{1}{1-\varepsilon/6} \tag{31}$$

These computations give an indication of how non-trivial values can be obtained for β and *v. The* formulae derived here are not exact, due to the severe simplifications made, but at least they show that β and *v* do not have to be 1/2 and in fact can have a complicated dependence on the dimension d.

A correct treatment is much more complex. Once $M_H(x)$ is not treated as a

constant, one could imagine expanding $M_H(x)$ in a Taylor's series about its value at some central location x_o relative to the location of the wavefunction $\psi(x)$, thus bringing in gradients of M_H. In addition, higher order terms in the expansion of the logarithm give higher powers of M_H. All this leads to a more complex form for the free energy functional F_L. with more gradient terms and more powers of M_H. The whole idea of the expansion in powers of M_H and powers of gradients can in fact be called into question. The fluctuations have an intrinsic size (i.e., m^2 has a size $\sim L^2$ as a consequence of the form of the integrand in Eq. 17) and it is not obvious that in the presence of these fluctuations, M is small. Since arbitrary wavelengths of fluctuations are important the function M is not sufficiently slowly varying tojustify an expansion in gradients either. This means that $F_L[M]$ could be an arbitrarily complicated function of M, an expression it is hard to write down, with thousands of parameters, instead of the simple Landau-Ginzburg form with only two parameters R_L and U_L.

Fortunately, the problem simplifies near 4 dimensions, due to the small magnitude of U_L, which is proportional to $\varepsilon = 4\text{-}d$. All the complications neglected above arise only to second order or higher in an expansion in U_L which means second order or higher in ε. The computations described here are exact to order ε. See Ref. 17.

The renormalization group approach that was defined in 1971 embraces both practical approximations leading to actual computations and a formalism.[17] The full formalism cannot be discussed here but the central idea of "fixed points" can be illustrated.

As the fluctuations on each length scale are integrated out a new free energy functional $F_{L+\delta L}$ is generated from the previous functional F_L. This process is repeated many times. If F_L and $F_{L+\delta L}$ are expressed in dimensionless form, then one finds that the transformation leading from F_L to $F_{L+\delta L}$ is repeated in identical form many times. (The transformation group thus generated is called the "renormalization group"). As L becomes large the free energy F_L approaches a fixed point of the transformation, and thereby becomes independent of details of the system at the atomic level. This leads to an explanation of the universality[18],, of critical behavior for different kinds of systems at the atomic level. Liquid-gas transitions, magnetic transitions, alloy transitions, etc. all show the same critical exponents experimentally; theoretically this can be understood from the hypothesis that the same "fixed point" interaction describes all these systems.

To demonstrate the fixed point form of the free energy functional, it must be put into dimensionless form. Lengths need to be expressed in units of L, and M, R_L, and U_L rewritten in dimensionless form. These changes are easily determined: write

$$x = Ly \tag{32}$$

$$M(x) = L^{1-d/2}m(y) \tag{33}$$

$$R_L = 1/L^2 r_L \tag{34}$$

$$U_L = L^{d-4}u_L \tag{35}$$

$$F_L = \int d^d y\{(\nabla m)^2 + r_L m^2(y) + u_L m^4(y)\} \tag{36}$$

The asymptotic solution for the dimensionless parameters r_L and u_L. is

$$r_L = cL^{2-\varepsilon/3} - \frac{\varepsilon}{3}\frac{1}{2-\varepsilon/3} \tag{37}$$

$$u_L = \frac{\varepsilon}{9} \tag{38}$$

Apart from the term in r_L, these dimensionless parameters are independent of L, denoting a free energy form which is also independent of L. The c term designates an instability of the fixed point, namely a departure from the fixed point which grows as L increases. The fixed point is reached only if the thermodynamic system is at the critical temperature for which c vanishes; any departure from the critical temperature triggers the instability.

For further analysis of the renormalization group formalism and its relation to general ideas about critical behavior, see e.g. ref. 17.

II I. *Some History Prior to 1971*

The first description of a critical point was the description of the liquid-vapor critical point developed by Van der Waals, [19] developed over a century ago following experiments of Andrews.[19] Then Weiss provided a description of the Curie point in a magnet."' Both the Van der Waals and Weiss theories were special cases of Landau's mean field theory." Even before 1900, experiments indicated discrepancies with mean Geld theory; in particular the experiments indicated that β was closer to 1/3 than 1/2.[19] In 1944, Onsager[6] published his famous solution to the two dimensional Ising model,[12] which explicity violated the mean field predictions. Onsager obtained $v = 1$ instead of the mean field prediction $v = 1/2$, for example. In the 1950's, Domb, Sykes, Fisher and others[21] studied simple models of critical phenomena in three dimensions with the help of high temperature series expansions carried to very high order, exacting critical point exponents by various extrapolation methods. They obtained exponents in disagreement with mean field theory but in reasonable agreement with experiment. Throughout the sixties a major experimental effort pinned down critical exponents and more generally provided a solid experimental basis for theoretical studies going beyond mean field theory. Experimentalists such as Voronel; Fairbanks, Buckingham, and Keller; Heller and Benedek; Ho and Litster, Kouvel and Rodbell, and Comly; Sengers; Lorentzen; Als-Nielsen and Dietrich; Birgeneau and Shirane; Rice; Chu; Teaney; Moldover; Wolf and Ahlers all contributed to this development, with M. Green, Fisher, Widom, and Kadanoff providing major coordination efforts." Theoretically, Widom[23] proposed a scaling law for the equation of state near the critical point that accommodated non-mean field exponents and predicted relations among them. The full set of scaling hypotheses were developed by Essam and Fisher, Domb and Hunter, Kadanoff, and Patashinskii, and Pokrovskii.[24] See also the inequalities of Rushbrooke [25] and Griffith.[26]

My own work began in quantum field theory, not statistical mechanics. A convenient starting point is the development ofrenormalization theory by Bethe, Schwinger, Tomonaga, Feynman, Dyson and others[27] in the late 1940's. The

first discussion of the "renormalization" group appeared in a paper by Stueckelberg and Petermann,[28] published in 1953.

In 1954 Murray Gell-Mann and Francis Low published a paper entitled "Quantum Electrodynamics at Small Distances"[29] which was the principal inspiration for my own work prior to Kadanoffs formulation[30] of the scaling hypothesis for critical phenomena in 1966.

Following the definition of Quantum Electrodynamics (QED) in the 1930's by Dirac, Fermi, Heisenberg, Pauli, Jordan, Wigner, et al.[27], the solution of QED was worked out as perturbation series in e_o, the "bare charge" of QED. The QED Lagrangian (or Hamiltonian) contains two parameters: e_o, and m_o, the latter being the "bare" mass of the electron. As stated in the introduction in QED the physical electron and photon have composite structure. In consequence of this structure the measured electric charge e and electron mass m are not identical to e_o and m_o, but rather are given by perturbation expansions in powers of e_o. Only in lowest order does one find $e = e_o$, and $m = m_o$. Unfortunately, it was found in the 30's that higher order corrections in the series for e and m are all infinite, due to integrations over momentum that diverge in the large momentum (or small distance) limit."

In the late 1940's renormalization theory was developed, which showed that the divergences of Quantum Electrodynamics could all be eliminated if a change of parametrization was made from the Lagrangian parameters e_o and m_o to the measurable quantities e and m, and if at the same time the electron and electromagnetic fields appearing in the Lagrangian were rescaled to insure that observable matrix elements (especially of the electromagnetic field) are finite.[27]

There are many reparametrizations of Quantum Electrodynamics that eliminate the divergences but use different finite quantities than e and m to replace e_o and m_o. Stueckelberg and Petermann observed that transformation groups could be defined which relate different reparametrizations - they called these groups "groupes de normalization" which is translated "renormalization group". The Gell-Mann and Low paper,[29] one year later but independently, presented a much deeper study of the significance of the ambiguity in the choice of reparametrization and the renormalization group connecting the difference choices of reparametrization. Gell-Mann and Low emphasized that e, measured in classical experiments, is a property of the very long distance behavior of QED (for example it can be measured using pith balls separated by centimeters, whereas the natural scale of QED is the Gompton wavelength of the electron, $\sim 10^{-11}$ cm). Gell-Mann and Low showed that a family ofalternative parameters e_λ could be introduced, any one of which could be used in place of e to replace e_o. The parameter e_λ is related to the behavior of QED at an arbitrary momentum scale λ instead of at very low momenta for which e is appropriate.

The family of parameters e_λ introduced by Gell-Mann and Low interpolate between the physical charge e and the bare charge e_o, namely e is obtained as the low momentum $(\lambda \to 0)$ limit of e_λ and e_o is obtained as the high momentum $(\lambda \to \infty)$ limit of e_λ.

Gell-Mann and Low found that e_λ^2 obeys a differential equation, of the form

$$\lambda^2 d(e_\lambda^2)/d(\lambda^2) = \psi(e_\lambda^2, m^2/\lambda^2) \quad (39)$$

where the φ function has a simple power series expansion with non-divergent coefficients independently of the value of λ, in fact as $\lambda \to \infty$ φ becomes a function of alone. This equation is the forerunner of my own renormalization group equations such as (22) and (23).

The main observation of Gell-Mann and Low was that despite the ordinary nature of the differential equation, Eq. (38), the solution was not ordinary, and in fact predicts that the physical charge e has divergences when expanded in powers of e_o, or vice versa. More generally, if e_λ is expanded in powers of the $e_{\lambda'}$, higher order coefficients contain powers of $\ell n(\lambda^2/\lambda'^2)$, and these coefficients diverge if either λ or λ' go to infinity, and are very large if λ^2/λ'^2 either very large or very small.

Furthermore, Gell-Mann and Low argued that, as a consequence of Eqn. (38), e_o, must have a fixed value independently of the value of e; the fixed value of e_o could be either finite or infinite.

When I entered graduate school at California Institute of Technology, in 1956, the default for the most promising students was to enter elementary particle theory, the field in which Murray Gell-Mann, Richard Feynman, and Jon Mathews were all engaged. I rebelled briefly against this default, spending a summer at the General Atomic Corp. working for Marshall Rosenbluth on plasma physics and talking with S. Chandrasekhar who was also at General Atomic for the summer. After about a month of work I was ordered to write up my results, as a result of which I swore to myself that I would choose a subject for research where it would take at least five years before I had anything worth writing about. Elementary particle theory seemed to offer the best prospects of meeting this criterion and I asked Murray for a problem to work on. He first suggested a topic in weak interactions of strongly interacting particles (K mesons, etc.) After a few months I got disgusted with trying to circumvent totally unknown consequences of strong interactions, and asked Murray to find me a problem dealing with strong interactions directly, since they seemed to be the bottleneck. Murray suggested I study K meson-nucleon scattering using the Low equation in the one meson approximation. I wasn't very impressed with the methods then in use to solve the Low equation, so I wound up fiddling with various methods to solve the simpler case of pion-nucleon scattering. Despite the fact that the one meson approximation was valid, if at all, only for low energies, I studied the high energy limit, and found that I could perform a "leading logarithms" sum very reminiscent of a very mysterious chapter in Bogoliubov and Shirkov's field theory text[31]; the chapter was on the renormalization group.

In 1960 I turned in a thesis to Cal Tech containing a mish-mash of curious calculations. I was already a Junior Fellow at Harvard. In 1962 I went to CERN for a year. During this period (1960-1963) I partly followed the fashions of the time. Fixed source meson theory (the basis for the Low equation) died, to be replaced by S matrix theory. I reinvented the "strip approximation" (Ter-Martirosyan had invented it first[32]) and studied the Amati-Fubini-Stanghellini

theory of multiple production.[33] I was attentive at seminars (the only period of my life when I was willing to stay fully awake in them) and I also pursued back waters such as the strong coupling approximation to fixed source meson theory?

By 1963 it was clear that the only subject I wanted to pursue was quantum field theory applied to strong interactions. I rejected S matrix theory because the equations of S matrix theory, even if one could write them down, were too complicated and inelegant to be a theory; in contrast the existence of a strong coupling approximation as well as a weak coupling approximation to fixed source meson theory helped me believe that quantum field theory might make sense. As far as strong interactions were concerned, all that one could say was that the theories one could write down, such as pseudoscalar meson theory, were obviously wrong. No one had any idea of a theory that could be correct. One could make these statements even though no one had the foggiest notion how to solve these theories in the strong coupling domain.

My very strong desire to work in quantum field did not seem likely to lead to quick publications; but I had already found out that I seemed to be able to get jobs even if I didn't publish anything so I did not worry about 'publish or perish' questions.

There was very little I could do in quantum field theory - there were very few people working in the subject, very few problems open for study. In the period 1963-1966 I had to clutch at straws. I thought about the "ξ-limiting" process of Lee and Yang." I spent a major effort disproving Ken Johnson's claims"' that he could define quantum electrodynamics for arbitrarily small e_o, in total contradiction to the result of Gell-Mann and Low. I listened to K. Hepp and others describe their results in axiomatic field theory[37]; I didn't understand what they said in detail but I got the message that I should think in position space rather than momentum space. I translated some of the work I had done on Feynman diagrams with some very large momenta (to disprove Ken Johnson's ideas) into position space and arrived at a short distance expansion for products of quantum field operators. I described a set of rules for this expansion in a preprint in 1964. I submitted the paper for publication; the referee suggested that the solution of the Thirring model might illustrate this expansion. Unfortunately, when I checked out the Thirring model, I found that while indeed there was a short distance expansion for the Thirring model,[38] my rules for how the coefficient functions behaved were all wrong, in the strong coupling domain. I put the preprint aside, awaiting resolution of the problem.

Having learned the fixed source meson theory as a graduate student, I continued to think about it. I applied my analysis of Feynman diagrams for some large momenta, to the fixed source model. I realized that the results I was getting became much clearer if I made a simplification of the fixed source model itself, in which the momentum space continuum was replaced by momentum slices.[39] That is, I rubbed out all momenta except well separated slices, e.g., $1 \leqslant |\mathbf{k}| \leqslant 2$, $\Lambda \leqslant |\mathbf{k}| \leqslant 2\Lambda, \Lambda^2 \leqslant |\mathbf{k}| \leqslant 2\Lambda^2, , \Lambda^n \leqslant |\mathbf{k}| \leqslant 2\Lambda^n$, etc. with A a large number.

This model could be solved by a perturbation theory very different from the methods previously used in field theory. The energy scales for each slice were

very different, namely of order Λ^n for the n^{th} slice. Hence the natural procedure was to treat the Hamiltonian for the largest momentum slice as the unperturbed Hamiltonian, and the terms for all lesser slices as the perturbation. In each slice the Hamiltonian contained both a free meson energy term and an interaction term, so this new perturbation method was neither a weak coupling nor a strong coupling perturbation.

I showed that the effect of this perturbation approach was that if one started with n momentum slices, and selected the ground state of the unperturbed Hamiltonian for the n^{th} slice, one wound up with an effective Hamiltonian for the remaining n-l slices. This new Hamiltonian was identical to the original Hamiltonian with only n-l slices kept, except that the meson-nucleon coupling constant g was renormalized (i.e., modified): the modification was a factor involving a non-trivial matrix element of the ground state of the n^{th}-slice Hamiltonian. [39]

This work was a real breakthrough for me. For the first time I had found a natural basis for renormalization group analysis: namely the solution and elimination of one momentum scale from the problem. There was still much to be done: but I was no longer grasping at straws. My ideas about renormalization were now reminiscent of Dyson's analysis of Quantum Electrodynamics."' Dyson argued that renormalization in Quantum Electrodynamics should be carried out by solving and eliminating high energies before solving low energies. I studied Dyson's papers carefully but was unable to make much use of his work."

Following this development, I thought very hard about the question "what is a field theory", using the ϕ^+ interaction of a scalar field (identical with the Landau-Ginzburg model of a critical point[14] discussed in my 1971 papers) as an example. Thoughout the '60's I taught quantum mechanics frequently, and I was very impressed by one's ability to understand simple quantum mechanical systems. The first step is a qualitative analysis minimizing the energy (defined by the Hamiltonian) using the uncertainty principle; the second step might be a variational calculation with wavefunctions constructed using the qualitative information from the first step; the final stage (for high accuracy) would be a numerical computation with a computer helping to achieve high precision. I felt that one ought to be able to understand a field theory the same way.

I realized that I had to think about the degrees of freedom that make up a field theory. The problem of solving the ϕ^4 theory was that kinetic term in the Hamiltonian (involving $(\nabla\phi)^2$) was diagonal only in terms of the Fourier components ϕ_k of the field, whereas the ϕ^+ term was diagonal only in terms of the field $\phi(x)$ itself. Therefore I looked for a compromise representation in which both the kinetic term and the interaction term would be at least roughly diagonal. I needed to expand the field $\phi(x)$ in terms of wavefunctions that would have minimum extent in both position space and momentum space, in other words wavefunctions occupying the minimum amount of volume in phase space. The uncertainty principle defines the lower bound for this volume, namely 1, in suitable units. I thought of phase space being divided up into blocks of unit volume. The momentum slice analysis indicated that momentum space should be marked off on a logarithmic scale, i.e. each momentum space volume should

correspond to a shell like the slices defined earlier, except that I couldn't leave out any momentum range so the shells had to be e.g...., $1 < |k| < 2$, $2 < |k| < 4$, etc. By translational invariance the position space blocks would all be the same size for a given momentum shell, and would define a simple lattice of blocks. The position space blocks would have different sizes for different momentum shells.

When I tried to study this Hamiltonian I didn't get very far. It was clear that the low momentum terms should be a perturbation relative to the high momentum terms but the details of the perturbative treatment became too complicated. Also my analysis was too crude to identify the physics of highly relativistic particles which should be contained in the Hamiltonian of the field theory."

However, I learned from this picture of the Hamiltonian that the Hamiltonian would have to be cutoff at some large but finite value of momentum k in order to make any sense out of it, and that once it was cutoff, I basically had a lattice theory to deal with, the lattice corresponding roughly to the position space blocks for the largest momentum scale. More precisely, the sensible procedure for defining the lattice theory was to define phase space cells covering all of the cutoff momentum space, in which case there would be a single set of position space blocks, which in turn defined a position space lattice on which the field ϕ would be defined. I saw from this that to understand quantum field theories I would have to understand quantum field theories on a lattice.

In thinking and trying out ideas about "what is a field theory" I found it very helpful to demand that a correctly formulated field theory should be soluble by computer, the same way an ordinary differential equation can be solved on a computer, namely with arbitrary accuracy in return for sufficient computing power. It was clear, in the '60's, that no such computing power was available in practice; all that I was able to actually carry out were some simple exercises involving free fields on a finite lattice.

In the summer of 1966 I spent a long time at Aspen. While there I carried out a promise I had made to myself while a graduate student, namely I worked through Onsager's solution of the two dimensional Ising model. I read it in translation, studying the field theoretic form given in Lieb, Mattis and Schultz.'"

When I entered graduate school, I had carried out the instructions given to me by my father and had knocked on both Murray Gell-Mann's and Feynman's doors, and asked them what they were currently doing. Murray wrote down the partition function for the three dimensional Ising model and said it would be nice if I could solve it (at least that is how I remember the conversation). Feynman's answer was "nothing". Later, Jon Mathews explained some of Feynman's tricks for reproducing the solution for the two dimensional Ising model. I didn't follow what Jon was saying, but that was when I made my promise. Sometime before going to Aspen, I was present when Ben Widom presented his scaling equation of state,[23] in a seminar at Cornell. I was puzzled by the absence of any theoretical basis for the form Widom wrote down; I was at that time completely ignorant of the background in critical phenomena that made Widom's work an important development.

As I worked through the paper of Mattis, Lieb, and Schultz, I realized there

should be applications of my renormalization group ideas to critical phenomena, and discussed this with some of the solid state physicists also at Aspen. I was informed that I had been scooped by Leo Kadanoff and should look at his preprint.[30]

Kadanoff's idea was that near the critial point one could think of blocks of magnetic moments, for example containing 2x2x2 atoms per block, which would act like a single effective moment, and these effective moments would have a simple nearest neighbor interaction like simple models of the original system. The only change would be that the system would have an effective temperature and external magnetic field that might be distinct from the original. More generally the effective moments would exist on a lattice of arbitrary spacing L times the original atomic spacing; Kadanoffs idea was that there would be L-dependent temperature and field variables T_L. and h_L, and that T_{2L}, and h_{2L}. would be analytic functions of T_L. and h_L. At the critical point, T_L, and h_L, would have fixed values independent of L. From this hypothesis Kadanoff was able to derive the scaling laws of Widom,[23] Fisher, etc.[24]

I now amalgamated my thinking about field theories on a lattice and critical phenomena. I learned about Euclidean (imaginary time) quantum field theory and the "transfer matrix" method for statistical mechanical models and found there was a close analogy between the two (see Ref. 17). I learned that for a field theory to be relativistic, the corresponding statistical mechanical theory had to have a large correlation length, i.e., be near a critical point. I studied Schiff's strong coupling approximation to the ϕ^4theory,[44] and found that he had ignored renormalization effects; when these were taken into account the strong coupling expansion was no longer so easy as he claimed. I thought about the implications of the scaling theory of Kadanoff, Widom et al. applied to quantum field theory, along with the scale invariance of the solution of the Thirring model" and the discussion of Kastrup and Mack of scale invariance in quantum field theory.[45] These ideas suggested that scale invariance would apply, at least at short distances, but that field operators would have non-trivial scale dimensions corresponding to the non-trivial exponents in critical phenomena. I redid my theory of short distance expansions based on these scaling ideas and published the result.[46] My theory did not seem to lit the main experimental ideas about short distance behavior (coming from Bjorken's and Feynman's analysis[47] of deep inelastic electron scattering) but I only felt confused about this problem and did not worry about it.

I returned to the fixed source theory and the momentum slice approximation. I made further simplifications on the model. Then I did the perturbative analysis more carefully. Since in real life the momentum slice separation factor Λ would be 2 instead of very large, the ratio $1/\Lambda$ of successive energy scales would be l/2 rather than very small, and an all orders perturbative treatment was required in $1/\Lambda$. When the lower energy scales were treated to all orders relative to the highest energy scale, an infinitely complicated effective Hamiltonian was generated, with an infinite set of coupling constants. Each time an energy scale was eliminated through a perturbative treatment, a new infinitely complicated Hamiltonian was generated. Nevertheless, I found that for sufficiently large Λ I could

mathematically control rigorously the effective Hamiltonians that were generated; despite the infinite number of couplings I was able to prove that the higher orders of perturbation theory had only a small and boundable impact on the effective Hamiltonians, even after arbitrarily many iterations.[48]

This work showed me that a renormalization group transformation, whose purpose was to eliminate an energy scale or a length scale or whatever from a problem, could produce an effective interaction with arbitrarily many coupling constants, without being a disaster. The renormalization group formalism based on fixed points could still be correct, and furthermore one could hope that only a small finite number of these couplings would be important for the qualitative behavior of the transformations, with the remaining couplings being important only for quantitative computations. In other words the couplings should have an order of importance, and for any desired but given degree of accuracy only a finite subset of the couplings would be needed. In my model the order of importance was determined by orders in the expansion in powers of $1/\Lambda$. I realized however that in the framework of an interaction on a lattice, especially for Ising-type models, locality would provide a natural order of importance - in any finite lattice volume there are only a finite number of Ising spin interactions that can be defined. I decided that Kadanoffs emphasis on the nearest neighbor coupling of the Ising model[30] should be restated: the nearest neighbor coupling would be the most important coupling because it is the most localized coupling one can define, but other couplings would be present also in Kadanoff's effective "block spin" Hamiltonians. A reasonable truncation procedure on these couplings would be to consider a finite region, say 3^3 or 4^3 lattice sites in size, and consider only multispin couplings that could fit into these regions (plus translations and rotations of these couplings).

Previously all the renormalization group transformations I was familiar with involved a fixed number of couplings: in the Gell-Mann-Low case just the electric charge e_λ, in Kadanoffs case an effective temperature and external field. I had tried many ways to try to derive transformationsjust for these fixed number ofcouplings, without success. Liberated from this restriction, it turned out to be easy to define renormalization group transformations; the hard problem was to find approximations to these transformations which would be computable in practice. Indeed a number renormalization group transformations now exist (see Section IV and its references).

In the fall of 1970 Ben Widom asked me to address his statistical mechanics seminar on the renormalization group. He was particularly interested because Di Castro and Jona-Lisinio had proposed applying the field theoretic renormalization group formalism to critical phenomena,[49] but no one in Widom's group could understand Di Castro and Jona-Lasinio's paper. In the course oflecturing on the general ideas of fixed points and the like I realized I would have to provide a computable example, even if it was not accurate or reliable. I applied the phase space cell analysis to the Landau-Ginzburg model of the critical point and tried to simplify it to the point of a calculable equation, making no demands for accuracy but simply trying to preserve the essence of the phase space cell picture. The result was a recursion formula in the form of a nonlinear integral trans-

formation on a function of one variable, which I was able to solve by iterating the transformation on a computer.[50] I was able to compute numbers for exponents from the recursion formula at the same time that I could show (at least in part) that it had a fixed point and that the scaling theory of critical phenomena of Widom et al. followed from the fixed point formalism. Two papers of 1971 on the renormalization group presented this work.[50]

Some months later I was showing Michael Fisher some numerical results from the recursion formula, when we realized, together, that the nontrivial fixed point I was studying became trivial at four dimensions and ought to be easy to study in the vicinity of four dimensions. The dimension d appeared in a simple way as a parameter in the recursion formula and working out the details was straightforward; Michael and I published a letter[51] with the results. It was almost immediately evident that the same analysis could be applied to the full Landau-Ginzberg model without the approximations that went into the recursion formula. Since the simplifying principle was the presence of a small coefficient of the ϕ^4 term, a Feynman diagram expansion was in order. I used my field theoretic training to crank out the diagrams and my understanding of the renormalization group fixed point formalism to determine how to make use of the diagrams I computed. The results were published in a second letter in early 1972.[52] The consequent explosion of research is discussed in Part IV.

There were independent efforts on the same area taking place while I completed my work. The connection between critical phenomena and quantum field theory was recognized by Gribov and Migdal and Polyakov[53] and by axiomatic field theorists such as Symanzik[54] T.T. Wu[55] worked on both field theory and the Ising model. Larkin and Khmelnitskii applied the field theoretic renormalization group of Gell-Mann and Low to critical phenomena in four dimensions and to the special case of uniaxial ferromagnets in three dimensions,[56] in both cases deriving logarithmic corrections to Landau's theory. Dyson formulated a somewhat artificial "hierarchical" model of a phase transition which was exactly solved by a one dimensional integral recursion formula.[54] This formula was almost identical to the one I wrote down later, in the 1971 paper. Anderson' worked out a simple but approximate procedure for eliminating momentum scales in the Kondo problem, anticipating my own work in the Kondo problem (see Sec. IV). Many solid state theorists were trying to apply diagrammatic expansions to critical phenomena, and Abe[58] and Scalapino and Ferrell[59] laid the basis for a diagrammatic treatment of models with a large number of degrees offreedom, for any dimension. (The limit of an infinite number of degrees of freedom had already been solved by Stanley[60]). Kadanoff was making extensive studies of the Ising model,[61] and discovered a short distance expansion for it similar to my own expansion for Geld theories. Fractional dimensions had been thought about before in critical phenomena.[62] Continuation of Feynman diagrams to non-integer dimensions was introduced into quantum Geld theory in order to provide a gauge invariant regularization procedure for non-abelian gauge theories:[63] this was done about simultaneously with its use to develop the ε expansion.

In the late '60's, Migdal and Polyakov[64] developed a "bootstrap" formulation

of critical phenomena based on a skeleton Feynman graph expansion, in which all parameters including the expansion parameter inself would be determined self-consistently. They were unable to solve the bootstrap equations because of their complexity, although after the ε expansion about four dimensions was discovered, Mack showed that the bootstrap could be solved to lowest order in ε[65]. If the 1971 renormalization group ideas had not been developed, the Migdal-Polyakov bootstrap would have been the most promising framework of its time for trying to further understand critical phenomena. However, the renormalization group methods have proved both easier to use and more versatile, and the bootstrap receives very little attention today.

In retrospect the bootstrap solved a problem I tried and failed to solve; namely how to derive the Gell-Mann-Low and Kadanoff dream of a fixed point involving only one or two couplings - there was only one coupling constant to be determined in the Migdal-Polyakov bootstrap. However, I found the bootstrap approach unacceptable because prior to the discovery of the ε expansion no formal argument was available to justify truncating the skeleton expansion to a finite number of terms. Also the skeleton diagrams were too complicated to test the truncation in practice by means of brute force computation of a large number of diagrams. Even today, as I review the problems that remain unsolved either by ε expansion or renormalization group methods, the problem of convergence of the skeleton expansion leaves me unenthusiastic about pursuing the bootstrap approach, although its convergence has never actually been tested. In the meantime, the Monte Carlo Renormalization group[66] has recently provided a framework for using small number of couplings in a reasonably effective and nonperturbative way: see Section IV.

I am not aware of any other independent work trying to understand the renormalization group from first principles as a means to solve field theory or critical phenomena one length scale at a time, or suggesting that the renormalization group should be formulated to allow arbitrarily many couplings to appear at intermediate stages of the analysis.

IV. *Results after 1971*

There was an explosion of activity after 1972 in both renormalization group and **E** expansion studies. To review everything that has taken place since 1972 would be hopeless. I have listed a number of review papers and books which provide more detailed information at the end of this paper. Some principal results and some thoughts for the future will be outlined here. The "ε expansion" about four dimensions gave reasonable qualitative results for three dimensional systems. It enabled a much greater variety of details of critical behavior to be studied than was previously possible beyond the mean field level. The principal critical point is characterized by two parameters: the dimension d and the number of internal components n. Great efforts were made to map out critical behavior as a function of d and n. ε expansion and related small coupling expansions were carried to very high orders by Brézin, Le Guillou, Zinn-Justin,[67] and Nickel"" led to precise results for d ≃ 3.[69,70] The large n limit and l/n expansion was pursued further." A new expansion in 2+ε dimensions was

developed for n>2 by Polyakov.[72] For n = 1 there is an expansion in 1+ε dimensions.[73] The full equation of state in the critical region was worked out in the ε expansion[74] and 1/n expansion.[75] The special case n = 0 was shown by De Gennes to describe the excluded volume problem in polymer configuration problems and random walks.[76] Corrections to scaling were first considered by Wegner[77]. A recent reference is Aharony and Ahlers.[78]

Besides the careful study of the principal critical point other types of critical points and critical behavior were pursued. Tricritical phenomena were investigated by Riedel and Wegner,[79] where Landau theory was found to breakdown starting in three dimensions instead of four.[80] More general multicritical points have been analyzed.[81] Effects of dipolar forces,[82] other long range forces,[83] cubic perturbations and anisotropies[84,85] were pursued. The problems of dynamics of critical behavior were extensively studied.[86] Liquid crystal transitions were studied by Halperin, Lubensky, and Ma.[87]

Great progress has been made in understanding special features of two dimensional critical points, even though two dimensions is too far from four for the ε expansion to be practical. The Mermin-Wagner theorem[88] foreshadowed the complex character of two dimensional order in the presence of continuous symmetries. The number ofexactly soluble models generalizing the Ising model steadily increases.[89] Kosterlitz and Thouless[90] blazed the way for renormalization group applications in two dimensional systems, following earlier work by Berezinskii.[91] They analyzed the transition to topological order in the 2-dimensional xy model with its peculiar critical point adjoining a critical line at lower temperatures; for further work see José et al[92] and Fröhlich and Spence[93,94]. Kadanoff and Brown have given an overview of how a number of the two-dimensional models interrelate.[95] A subject of burning recent interest is the two-dimensional melting transition.[96] Among generalizations of the Ising model, the 3 and 4 state Potts model have received special attention. The three-state Potts model has only a first order transition in mean field theory and an expansion in 6-ε dimensions but has a second order transition in two dimensions.[97] The four state Potts model has exceptional behavior in two dimensions (due to a "marginal variable"), which provides a severe challenge to approximate renormalization methods. Notable progress on this model has been made recently.[98]

A whole vast area of study concerns critical behavior or ordering in random systems, such as dilute magnets, spin glasses, and systems with random external fields. Random systems have qualitative characteristics of a normal system in two higher dimensions as was discovered by Lacour-Gayet and Toulouse[99] Imry, Ma, Grinstein, Aharony,[100] and Young"" and confirmed by Parisi and Sourlas[102] in a remarkable paper applying 'supersymmetry' ideas from quantum field theory.[103] The "replica method" heavily used in the study of random systems[104] involves an $n \to 0$ limit, where n is the number ofreplicas similar to the De Gennes $n \to 0$ limit defining random walks.[76] There are serious unanswered questions surrounding this limiting process. Another curious discovery is the existence of an $\varepsilon^{1/2}$ expansion found by Khmelnitskii and Grinstein and Luther.[105]

Further major areas for renormalization group applications have been in

percolation,[106] electron localization or conduction in random media, [107] the problems of structural transitions and "Lifshitz" critical points,[108] and the problem of interfaces between two phases.[109]

Much of the work on the ε expansion involved purely Feynman graph techniques; the high order computations involved the Callan-Symanzik formulation[110] of Gell-Mann Low theory. The computations also depended on the special diagram computation techniques of Nickel[68] and approximate formulae for very large orders of perturbation theory first discussed by Lipatov.[111] In lowest order other diagrammatic techniques also worked, for example the Migdal-Polyakov bootstrap was solved to order ε by Mack.[65]

The modern renormalization group has also developed considerably, Wegner[77,112,113] strengthened the renormalization group formalism considerably. A number ofstudies, practical and formal[114] were based on the approximate recursion formula introduced in 1971. Migdal and Kadanoff[115] developed an alternative approximate recursion formula (based on "bond moving" techniques). Real space renormalization group methods were initiated by Niemeijer and Van Leeuwen[116] and have been extensively developed since.[117, 118] The simplest real space transformation is Kadanoff's "spin decimation" transformation[119].[120] where roughly speaking some spins are held fixed while other spins are summed over, producing an effective interaction on the fixed spins.

The decimation method was very successful in two dimensions where the spins on alternative diagonals of a square lattice were held fixed.[120] Other real space formulations[116, 117] involved kernels defining block spin variables related to sums of spins in a block (the block could be a triangle, square, cube, a lattice site plus all its nearest neighbors, or whatever).

Many of the early applications of real space renormalization group methods gave haphazard results - sometimes spectacularly good, sometimes useless. One could not apply these methods to a totally new problem with any confidence of success. The trouble was the severe truncations usually applied to set up a practical calculation; interactions which in principle contained thousands of parameters were truncated to a handful of parameters. In addition, where hundreds of degrees of freedom should be summed over (or integrated over) to execute the real space transformation, a very much simplified computation would be substituted. A notable exception is the exactly soluble differential renormalization group transformation of Hilhorst, Schick and Van Leeuwen, which unfortunately can be derived only for a few two dimensional models.[121, 122]

Two general methods have emerged which do not involve severe truncations and the related unreliability. First of all, I carried out a brute force calculation for the two dimensional Ising model using the Kadanoff decimation approach[119, 120] (as generalized by Kadanoff). Many interaction parameters (418) were kept and the spin sums were carried out over a very large finite lattice. The results were very accurate and completely confirmed my hypothesis that the local couplings of the shortest range were the most important. Most importantly the results could be an *optimization principle.* The fixed point of Kadanoff's decimation transformation depends on a single arbitrary parameter; it was possible to determine a best value for this parameter from internal consistency consider-

ations. Complex calculations with potentially serious errors always are most effective when an optimization principle is available and parameters exist to optimize on.[123] This research has never been followed up, as is often the case when large scale computing is involved. More recently, the Monte Carlo Renormalization group method,[66] developed by Swendsen, myself, Shenker, and Tobochnik (see also Hilhorst and Van Leeuwen)"' has proved very accurate and may shortly overtake both the high temperature expansions and the ε expansion as the most accurate source of data on the three dimensional Ising model. The Monte Carlo Renormalization Group is currently most successful on two dimensional problems where computing requirements are less severe: it has been applied successfully to tricritical models and the four-state Potts model.[124] In contrast, the ε expansion is all but useless for two dimensional problems. Unfortunately, none of the real space methods as yet provide the detailed information about correlation functions and the like that are easily derived in the ε expansion.

A serious problem with the renormalization group transformations (real space or otherwise) is that there is no guarantee that they will exhibit fixed points. Bell and myself[125] and Wegner in a more general and elegant way[113] have shown that for some renormalization group transformations, iteration of a critical point does not lead to a fixed point, presumably yielding instead interactions with increasingly long range forces. There is no known principle for avoiding this possibility, and as Kadanoff has showed using his decimation procedure,[120] a simple approximation to a transformation can misleadingly give a fixed point even when the full transformation cannot. The treatment that I gave of the two dimensional Ising model has self consistency checks that signal immediately when long range forces outside the 418 interactions kept are becoming important. Nothing is known yet about how the absence of a fixed point would be manifested in the Monte Carlo renormalization group computations. Cautions about real space renormalization group methods have also been advanced by Griffiths et al.[126]

There is a murky connection between scaling ideas in critical phenomena and Mandelbrot's "fractals" theory - a theory of scaling of irregular geometrical structures (such as coastlines).[127]

Renormalization group methods have been applied to areas other than critical phenomena. The Kondo problem is one example. Early renormalization group work was by Anderson" and Fowler and Zawadowski.[128] I then carried out a very careful renormalization group analysis of the Kondo Hamiltonian,[129] producing effective Hamiltonians with many couplings for progressively smaller energy scales, following almost exactly the prescription I learned for fixed source meson theory. The result was the zero-temperature susceptibility to about 1 % accuracy, which was subsequently confirmed by Andrei and Wiegmann's[7] exact solution. Renormalization group methods have been applied to other Hamiltonian problems, mostly one dimensional.[130] In multidimensional systems and in many one dimensional systems, the effective Hamiltonians presently involve too many states to be manageable.

The renormalization group has played a key role in the development of Quan-

turn Chromodynamics - the current theory of quarks and nuclear forces. The original Gell-Mann Low theory[29] and the variant due to Callan and Symanzik[110] was used by Politzer, Gross and Wilczek[131] to show that nonabelian gauge theories are asymptotically free. This means that the short distance couplings are weak but increase as the length scale increases; it is now clear that this is the only sensible framework which can explain, qualitatively, the weak coupling that is evident in the analysis of deep inelastic electron scattering results (off protons and neutrons) and the strong coupling which is evident in the binding of quarks to form protons, neutrons, mesons, etc.[132] I should have anticipated the idea of asymptotic freedom[133] but did not do so. Unfortunately, it has been hard to study quantum chromodynamics in detail because of the effects of the strong binding of quarks at nuclear distances, which cannot be treated by diagrammatic methods. The development of the lattice gauge theory by Polyakov and myself[134] following pioneering work of Wegner[135] has made possible the use of a variety of lattice methods on the problems of quantum chromodynamics,[136] including strong coupling expansions, Monte Carlo simulations, and the Monte Carlo renormalization group methods.[67], [137] As computers become more powerful I expect there will be more emphasis on various modern renormalization group methods in these lattice studies, in order to take accurately into account the crossover from weak coupling at short distances to strong coupling at nuclear distances.

The study of unified theories ofstrong, weak and electromagnetic interactions makes heavy use of the renormalization group viewpoint. At laboratory energies the coupling strengths of the strong and electromagnetic interactions are too disparate to be unified easily. Instead, a unification energy scale is postulated at roughly 10^{15} GeV; in between renormalization group equations cause the strong and electroweak couplings to approach each other, making unification possible. Many grand unified theories posit important energy scales in the region between 1 and 10^{15} GeV. It is essential to think about these theories one energy scale at a time to help sort out the wide range of phenomena that are predicted in these theories. See Langacker[138] for a review. The study of grand unification has made it clear that Lagrangians describing laboratory energies are phenomenological rather than fundamental, and this continues to be the case through the grand unification scale, until scales are reached where quantum gravity is important. It has been evident for a long time that there should be applications of the renormalization group to turbulence, but not much success has been achieved yet. Feigenbaum[138] developed a renormalization group-like treatment of the conversion from order to chaos in some simple dynamical systems,[140] and this work may have applications to the onset of turbulence. Feigenbaum's method is probably too specialized to be of broader use, but dynamical systems may be a good starting point for developing more broadly based renormalization group methods applicable to classical partial differential equations.[141]

In my view the extensive research that has already been carried out using the renormalization group and the ε expansion is only the beginning of the study of a much larger range of applications that will be discovered over the next twenty years (or perhaps the next century will be required). The quick successes

of the ε expansion are now past, and I believe progress now will depend rather on the more difficult, more painful exercises such as my own computations on the two dimensional Ising model and the Kondo problem,[120] or the Monte Carlo Renormalization group[66] computations. Often these highly quantitative, demanding computations will have to precede simpler qualitative analysis in order to be certain the many traps potentially awaiting any renormalization group analysis have been avoided.

Important potential areas of application include the theory of the chemical bond, where an effective interaction describing molecules at the bond level is desperately needed to replace current ab initio computations starting at the individual electron level.[142] A method for understanding high energy or large momentum transfer Quantum Chromodynamics (QCD) cross sections (including non-perturbative effects) is needed which will enable large QCD backgrounds to be computed accurately and subtracted away from experimental results intended to reveal smaller non-QCD effects. Practical areas like percolation, frost heaving, crack propagation in metals, and the metallurgical quench all involve very complex microscopic physics underlying macroscopic effects, and most likely yield a mixture of some problems exhibiting fluctuations on all length scales and other problems which become simpler classical problems without fluctuations in larger scales.

I conclude with some general references. Two semi-popular articles on the renormalization group are Wilson (1979) and Wilson (1975). Books include Domb and Green (1976); Pfeuty and Toulouse (1977); Ma (1976); Amit (1978); Patashinskii and Pokrovskii (1979)[143]; and Stanley (1971)[144] Review articles and conference proceedings include Widom (1975)[145]; Wilson and Kogut (1974); Wilson (1975); Fisher (1974)[146]; Wallace and Zia (1978); Greer and Moldover (1981); and Lévy et al. (1980).[147]

I thank the National Science Foundation for providing funding to me, first as a graduate student, then throughout most of my research career. The generous and long term commitment of the United States to basic research was essential to my own success. I thank my many colleagues at Cornell, especially Michael Fisher and Ben Widom, for encouragement and support. I am grateful for the opportunity to be a member of the international science community during two decades of extraordinary discoveries.

REFERENCES

1. Wilson, K. G. Physica 73, 119 (1974).
2. See, e.g. Rose H. A. and Sulem, P. L.J. Physique 39, 441 (1978).
3. See, c.g. Criegee L. and Knies, G. Phys. Rcpts. 83, 151 (1982).
4. See, CRC Handbook of chemistry and Physics (62nd Edition), Weast, R. C. rd. (CRC Press, Boca Raton, Florida, 1981), p. F-76.
5. See, e.g., Anderson, P. W . J. Phys. C3, 2346 (1970).
6. Onsager, L. Phys. Rev. 6.5, 117 (1944).
7. Andrei, N. Phys. Rev. Lett. 45, 379 (1980; Phys. Lett. 87A, 299 (1982); Andrei, N. and Lowcnstein, J, H. Phys. Rev. Lett. 46, 356 (1981): Wiegmann, P. B. Phys. Lett. 80A, 163 (1980); J. Phys. C14, 1463 (1981); Filyov V. M. Tzvrlik, A. M. and Weigmann, P. B. Phys. Lett. *81A,* 175 (1981).
8. Johnson', K. Nuovo Cimcnto 20, 773 (1961).
9. The experimcntal measurements on fluids (e.g., SF_6, He^3, and various organic fluids.) give $\beta = 0.32 \pm 0.02$, while current theoretical computations give $\beta = 0.325 \pm 0.005$: see Greer S. C and Moldover, M. R. Annual Rcview of Physical Chemistry 32, 233 (1981) for data and caveats.
10. For experimental reviews, see e.g. Heller, P. Repts. Prog. Phys. 311, 731 (1967): Kadanoff, I.. Götze, W. Hamblen, D. Hecht, R. Lewis, E. A. S. Palciauskas, V. V. Rayl. M. Swift. ,J. Aspres, D. Kane, ,J. *Static Phenomena* near *Critical Points: Theory: and Experiment,* Rev. Mod. Phys. 39, 395 (1967).
11. For the alloy transition in β brass see Als-Nielsen, J in Domb. C. and Green, M. S. eds.. *Phase Transitions and Critical Phenomena.* Vol. 5a (Academic Press, *New* York. 1978) p.87. See Ref. 10 for carlier reviews of other systems.
12. For a history of the Lcnz-Ising model see Brush. S. G Rev. Mod. Phys. 39, 883 (1967).
13. Landau. L. D. Phys. Zurn. Sowjetunion *11, 545* (1937); English translation: ter Haar, D. Men *of Physics: I.. D. Landau. Vol II.* II (Pergamon, Oxford. 1969).
14. Ginzburg. V. L. and Landau. L. D. Zh. Eksperim. i Teor. Fiz. 20. 1061 (1950). See also Schrieffer, J. Superconductivity [Benjamin. New York. 1964) p.19.
15. Ginzburg, V.L 1.. Fiz. Tverd. Tela 2 2031 (1960). English translation: Soviet Physics: Solid St. 2 1824 (1960).
16. Wilson, K. G. and Fisher. M. E. Phys. Rev. Letters 28. 240 (1972).
17. Wilson. K G and Kogut. J. Phys. Repts. 12C. 75. 7.5 (1974).
18. See, e.g., Guggcnhcim. E. A.J. Chem. Phys 13, 253 (1945): Griffiths. R. B. Phys. Rev. Letts. 24. 1479 (1970): Griffiths R. B. and Whcelcr, J. B. Phys. Rev. A2 1047 (1970): Kadanoff L. in *Phase Transitions and Critical Phenomena.* Domb C. and Green. M. S. Eds. Vol. *5a* (Academic. New York, 1978, p. I .).
19. See, e.g., DeBoer, J. Physica 7.7. 1 (1974): Klein, M.D. Physica 73, 28 (1971); Lcvelt.]. 51. H. Sengers, Physica 7.7, 73 (1974).
20. Weiss. P. J. Phys. 6, 661 (1907).
21. Sec. Domb. C. Proc. Roy. Soc. *A199,* 199 (1949) and, e.g. Fisher. M.E. E. Reports. Prog. Phys. 30, 615 (1967).
22. See. e.g., ref: 10 and Ahlers, G. Revs. Mod. Phys. 52. 489 (1980).
23. Wisdom, B. J. Chcm. Phys. 43, 3898 (1965).
24. Essam J. W. and Fisher, M. E. J. Chem. Phys. 38, 802 (1963): Domb C. and Hunter. D. L. Proc. Phys. Soc. 86, 1147 (1965) Kadanoff, L. P. Physics, 2. 263 (1966): Patashiniskii A. Z. and Prokrovskii. V. L. Zh. Eksper. Teor. Fiz. 50. 439 (1966); English translation: Soviet Physics-JETP 23,. 292 (1966).
25. Rushbrookc. G. S.. 1963. J. Chcm. Phys. 39. 842.
26. Griffiths, R. B.. 1965. Phys. Rev. Lett. 14, 623.
27. See the reprint collection Schwinger, J. Ed Quantum Electrodynamics.. Dover. New York. 1958).
28. Stueckelberg E. C. G. and Petermann. A Hclv. Phys. Acta 26. 499 (1953): see also Pctermann. A. Phys. Rcpts. 53, I57 11979).
29. Gell-Mann M. and Low. F. E. Phys. Rev. 95. 1300 (1954).

30. Kadanoff, L. P. Physics 2, 263 (1966).
31. Bogoliubov N. N. and Shirkov, D. V. Introduction to the *Theory of Quantized Fields* (Interscience; New York, 1959), Ch. VIII.
32. Ter-Martirosyan, K. A. Zh. Eksperim. Teor. Fiz. 39, 827 (1960); Soviet Physics-JETP 12, *575* (1961).
33. See, e.g. Wilson, K. G. Acta Physica Austriaca 17, 37 (1963-64) and references cited therein.
34. Wenzel, G. Helv. Phys. Acta 13, 269 (1940); 14, 633 (1941); see also Henley E. M. and Thirring, W. *Elementary Quantum Field Theory* (McGraw Hill, New York, 1962).
35. Lee T. D. and Yang, C. N. Phys. Rev. 128,885 (1962).
36. Johnson, K. Baker, M. and Willey, R. S. Phys. Rev. Lett. 11, 518 (1963). For a subsequent view see Baker, M. and Johnson, K., Phys. Rev. 183, 1292 (1969); ibid D3,2516,2541 (1971).
37. See e.g. Hepp, K. Acta Physica Austriaca 17, 85 (1963-64).
38. See e.g. Lowenstein, J. H. Commun. Math. Phys. 16, 265 (1970); Wilson, K. G. Phys. Rev. D2, 1473 (1970).
39. Wilson, K. G. Phys. Rev. 140, B445 (1965).
40. Dyson, F. J. Phys. Rev. 83, 608, 1207 (1951).
41. Mitter, P. K., and G. Valent, 1977, Phys. Lett. *B70, 65.*
42. See, e.g. Kogut J. and Susskind, L. Phys. Repts. 8C, 75 (1973) and referenccs cited therein.
43. Schultz, T. D. Mattis, D. C. and Lieb, E. H. Rev. Mod. Phys. 36, 856 (1964).
44. Schiff, L. I. Phys. Rev. 92, 766 (1953).
45. See, e.g. Mack, G. Nucl. Phys. *B5,* 499 (1968) and references cited therein.
46. Wilson, K. G. Phys. Rev. 179, 1499 (1969).
47. Bjorken, J. Phys. Rev. 179, 1547 (1969); Feynman, R. P. Phys. Rev. Lett. 23, 1415 (1969): for a review see Yan, T. M. Annual Review of Nuclear Science, 26, 199 (1976) and Feynman, R. P. *Photon-Hadron Interactions,* Benjamin, W. A. (Reading, Mass, 1972).
48. Wilson, K. G. Phys. Rev. D2, 1438 (1970).
49. Di Castro, C. and Jona-Lasinio, G. Phys. Letters *29A, 322* (1969).
50. Wilson, K. G Phys. Rev. *B4, 3174,* 3184 (1971).
51. Wilson, K. G. and Fisher, M. E. Phys. Rev. Lett. 28, 240 (1972).
52. Wilson. K. G. Phys. Rev. Lett. 28, 548 (1972).
53. Gribov, V. N. and Migdal, A. A. Zh. Eksp. Teor. Fiz. 5.5, 1498 (1968); English Translation: Soviet Physics-JETP 28, 784 (1968); Migdal, A. A ibid. 59, 1015 (1970); JETP 32, 552 (1971); Polyakov, A. M. ibid. 55, 1026 (1968); JETP 28, 533 (1969): Polyakov, A.M . ibid. 57, 271 (1969); JETP 30, 151 (1970); Polyakov, A. M. ibid. 59, 542 (1970); JETP 32. 296 (1971).
54. Symanzik, K.J. Math. Phys. 7, 510 (1966).
55. Wu, T. T. Phys. Rev. 149, 380 (1966); McCoy, B. M. and Wu, T. T. *The Two Dimensional Ising Model* (Harvard University Press, Cambridge, 1973); McCoy, B. M. Tracy, C. and Wu, 'I'. T. Phys. Lett. *MA, 283 (1977).*
56. Larkin, A. I. and Khmelnitskii, D. E. Zh. Eksp. Teor. Fiz. .56, 2087 (1969); English Translation: Soviet Physics JETP 29, 1 1123 (1969).
57. Dyson, F.J. Commun. Math. Phys. 12, 91 (1969); see also Baker. G. A. Jr., Phys. Rev. *B5, 2622 (1972).*
58. Abe, R. Progr. Theor. Phys. (Kyoto) 48, 1414 (1972); ibid. 49, 113 (1973); Abe, R. and Hikami, S. Phys. Letters *45A,* 11 *(1973)* Hikami, S. Progr. Theor. Phys. (Kyoto) 49, 1096 (1973).
59. Ferrell, R. A. and Scalapino, D.J. Phys. Rev. Letters 29, 413 (1972): Phys. Letters *41A.* 371 (1972).
60. Stanley, H. E. Phys. Rev. 176, 718 (1968).
61. Kadanoff L. P. Phys. Rev. Lett. 23, 1430 (1969); Phys. Rev. 188, 859 (1969).
62. See, e.g. Fisher, M. E. and Gaunt, D. S. Phys. Rev. 133, A224 (1964); Widom, B. Mol. Phys. *25, 657* (1973).
63. 't Hooft, G. and Veltman, M. Nucl. Phys. *B44. 189* (1972); Bollini, C. G and Giambiagi, J.J. Phys. Letts. *40B, 566 (1972);* Ashmore, J. F. Lettere Nuovo Cimento 4. 289 (1972).
64. Patashinskii, A. Z. and Pokrovskii, V. I.. Zh. Eksp. Teor. Fiz. 46, 994 (1964): English Translation: Soviet Physics, JETP /9, 677 (1964); Polyakov, A. M Zh. ETF. Pis. Red. *12, 538*

(1970); English Translation: JETP Letters 12, 381 (1970); Migdal, A. A. Phys. Letts. *37B, 386* (1971); Mack, G. and Symanzik, K. Common Math Phys. 27,247 (1972) and references cited therein.

65. Mack, G. in Strong Interaction Physics, Rühl, W. and Vanira, A. eds. (Springer-Verlag, Berlin, 1973), p.300.
66. Swendsen, R. H. Phys. Rev. Lett. 42, 859 (1979); Phys. Rev. *B20,* 2080 (1979); Wilson, K. G. Monte Carlo Calculation for the Lattice Gauge Theory, in *Recent Developments in Gauge Theories,* 't Hooft, G. et al. Eds. (Plenum Press, New York, 1980); Shenker, S. and Tobochnik, J. Phys. Rev. *B22, 4462 (* 1980).
67. See, e.g. the review of Zinn-Justin, J. Phys. Repts. 70, 109 (1981); see further Nickel, B. Physica *106A, 48* (1981).
68. Nickel, B. (unpublished), cited in Ref. 64 and prior papers.
69. Parisi, G., 1980, J. Stat. Phys. 23, 49.
70. Vladimirov, A. A., D. I. Kazakov, and O. Tarasov, 1979, Sov. Phys. JETP *50,* 521.
71. See, e.g. Ma, S. K. in Domb, C. and Green, *M. S.* eds. *Phase Transitions and Critical Phenomena,* Vol. 6 (Academic Press, London, 1976), p.250, and references cited therein.
72. Polyakov, A. M. Phys. Lett. *59B,* 79 (1975); Migdal, A. A., Zh. Eksp. Tear. Fiz. 69,. 1457 (1975); English translation: Soviet Physics-JETP 42, 743 (1976); Brkzin, E. Zinn-Justin, ,J. and LeGuillou, J, C. Phys. Rev. *B14,* 4976, (1976); Bhattacharjee, J. K. Cardy, J. L. and Scalapino, D. J. Phys. Rev. *B25,* 1681 (1982).
73. Wallace, D. J. and Zia, R. K. Phys. Rev. Lett. 35, 1399 (1979); Widom, B. (ref. 59).
74. Brézin, E. Wallace, D. and Wilson, K. G. Phys. Rev. Letters, 29, 591 (1972); Phys. Rev. *B7,* 232 (1973); Avdeeva, G. M. and Migdal, A. A. Zh. ETF. Pis. Red. 16, 253 (1972); English Translation: Soviet Physics-JETP Letters 16, 178 (1972); Avdeeva, G. M. Zh. Eksp. Tear. Fiz. 64, 741 (1973), English Translation: Soviet Physics-JETP 37, 377 (1973).
75. Brézin, E. and Wallace, D. J. Phys. Rev. *B7, 1967 (1973);* for a review see Ma S.-K. in Domb, C. and Green, M. S. Eds., Phase Transitions and Critical Phenomena, Vol. 6 (Academic Press, London, 1976) p.250.
76. DeGennes, P. G. Phys. Letters *38A, 339* (1972); des Cloiseaux, J. J. Phys. (Paris) 36, 281 (1975).
77. Wegner, F. J., 1972a, Phys. Rev. *B5, 4529.*
78. Aharony, A. and G. Ahlers, 1980, Phys. Rev. Lett. 44, 782.
79. Riedel, E. K. and Wegner, F. J. Phys. Rev. Lett. 29, 349 (1972); Phys. Rev. *B7, 248* (1973).
80. Stephen, M. J., E. Abrahams, and J. P. Straley, 1975, Phys. Rev. *B12, 256.*
81. See, e.g., Fisher, M. E. in A.I.P. Conference Proceedings #24, Magnetism *and Magnetic Materials* 1974, p. 273.
82. Fisher, M. E. and Aharony, A. Phys. Rev. Lett. 30, 559 (1973); Aharony, A. and Fisher, M. E. Phys. Rev. *B8,* 3323 (1973); Aharony, A. Phys. Rev. *B8, 3342, 3349, 3358, 3363* (1973); Aharony, A. Phys. Letters *44A,* 313 (1973).
83. Suzuki, M. Phys. Letters, *42A, 5* (1972); Suzuki, M. Progr. Theor. Phys. (Kyoto) 49, 424, 1106, 1440 (1973); Fisher, M. E. Ma, S. and Nickel, B. G. Phys. Rev. Letters 29, 917 (1972); Baker Jr., G. A. and Golner, G. R. Phys. Rev. Letters 31, 22 (1973); Suzuki, M. Yamazaki, Y. and Igarishi, G. Phys. Letters *42A,* 313 (1972); Sak, J. Phys. Rev. *B8, 281 (1973).*
84. Pfeuty, P. and Fisher, M. E. Phys. Rev. *B6,* 1889 (1972); Wallace, D. J. J. Phys. C6, 1390 (1973); Ketley, L. J. and Wallace, D. J. J. Phys. *A6,* 1667 (1973); Aharony, A. Phys. Rev. Letters 31, 1494 (1973); Suzuki, M. Progr. Theor. Phys. (Kyoto) 49, 1451 (1973); Liu, I.. Phys. Rev. Letters 31, 459 (1973); Grover, M. K. Phys. Letters *44A,* 253 (1973); Chang, T. S. and Stanley, H. E. Phys. Rev. *B8, 4435 (1973);* Aharony, A. Phys. Rev. *B8, 4270* (1973); A recent reference is Blankschtein, D. and Mukamel, D. Phys. Rev. *B25, 6939* (1982).
85. Wegner, F. J., 1972b, Phys. Rev. *B6,* 1891.
86. Halperin, B. I. Hohenberg, P. C. and Ma, S.-K. Phys. Rev. Letters 29, 1548 (1972); Suzuki, M. and Igarishi, G. Progr. Theor. Phys. (Kyoto) 49, 1070 (1973): Suzuki, M. Phys. Letters *43A, 245* (1973); Suzuki, M. Progr. Theor. Phys. (Kyoto) 50, 1767 (1973); for a review see Hohenberg, P. C. and Halperin, B. I. Revs. Mod. Phys. 49, 435 (1977); for a recent Monte Carlo Renormalization Group method see Tobochnik, J, Sarker, S. and Cordery, R. Phys.

Rev. Lett, 46, 1417, (1981). Other recent references are Ahlers, G. Hohenberg, P. C. and Kornblit, A. Phys. Rev. *B25,* 3136 (1982); Heilig, S. J. Luscombe, J. Mazrnko, G. F. Oguz, E. and Valls, 0. T. Phys. Rev. *B25, 7003 (1982).*

87. Halperin B. I., T. C. Lubensky, and S.-k. Ma, 1974, Phys. Rev. Letters, 32, 242.
88. Mermin, N. D. and Wagner, H. Phys. Rev. Lett. 17, 1133 (1966); Mermin, N. D. J. Math. Phys. 8, 1061 (1967); Hohenberg, P. C. Phys. Rev. 158, 383 (1967).
89. See, e.g., Baxter, R. *Exactly Solved Models in Statistical Mechanics,* (Academic Press, New York, 1982).
90. Kosterlitz,J. M. and Thouless, D. J. J. Phys. C6, 1181 (1973); Koster1itz.J. M. ,J. Phys. C7, 1046 (1974); for a review see Kosterlitz, J. M. and Thouless, D. J. Prog. Low Temp. Phys. 7, 373 (1978).
91. Bcrezinskii, V. L. Zh. Eksp. Tear. Fiz. 59, 907 (1970); (Soviet Physics-JETP 32,493 (1971)); Berezinskii, V. I.. Zh. Eksp. Tear. Fiz. *61,* 1144 (1971) (Soviet Physics-JETP 34,610 (1972)).
92. José, J. V., L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, 1977, Phys. Rev. *B16, 1217.*
93. Frölich, J., and T. Spencer, 1981, Phys. Rev. Lett. 46. 1006.
94. Fröhlich, J., and T. Spencer, 1981, Commun. Math. Phys. *81, 527.*
95. Kadanoff, L. P. and Brown. A. C. Annals Phys. (N.Y.) 121, 318 (1979).
96. Nelson, D. R. and Halperin, B. I. Phys. Rev. *B19, 2457 (1979);* Young, A. P. Phys. Rev. *B19,* 1855 (1979).
97. Baxter, R. J. J. Phys. C6, 2445 (1973) rigorouus 2-d solution); see e.g., Banavar, J. R. Grest, G. S and Jasnow, D. Phys. Rev. *B25, 4639 (1982)* and references cited therin (d>2).
98. Nienhuis, B. Berker, A. N. Riedel, E. K. and Schick, M. Phys. Rev. Lett. 43. 737 (1979); Swendsen, R. H. Andelman, D. and Berker, A. N. Phys. Rev. *B24,6732* (1982) and references cited therein.
99. Lacour-Gayet, P., and G. Toulouse, 1974, J. Phys. 35. 425.
100. Imry, Y. and Ma, S.-K. Phys. Rev. Lett. 35, 1399 (1975); Grinstein, G. Phys. Rev. Lctt. 37, 944 (1976); Aharony, A. Imry, Y. and Ma, S.-K. Phys. Rev. Lett. 37. 1367 (1976); a recent reference is: Mukamel, D. and Grinstein, G. Phys. Rev. *B25, 381 (1982).*
101. Young, A. P., 1977, J. Phys. *C10,* L257.
102. Parisi, G. and Sourlas, N. Phys. Rev. Lctt. 43, 744 (1979).
103. Parisi, G., and N. Sourlas, 1981, Phys. Rev. Lett. 46. 871.
104. Edwards, S. F. and Anderson, P. W. J. Phys. F5, 965 (1975).
105. Khmelnitskii, D. E. Zh. Eksp. Teor. Fiz. 68, 1960 (1975), (Sov. Phys.-JETP *41. 981 (1976)):* Grinstein, G. and Luther, A. Phys. Rev. *B13, 1329 (1976).*
106. See, e.g., the review of Essam, J. W. Repts. Prog. Phys. 43. 833 (1980): a recent reference is Lobb, C. J. and Karasek, K. R. Phys. Rev. *B25, 492 (1982).*
107. Nagaoka, Y., and H. Fukuyama, editors, 1982, *Anderson Localization* Springer Series in Solid-State Science (Springer-Berlin), Vol. 39.
108. See, e.g., the review of Bruce, A. D. Adv. Phys. 29, 111 (1980); a recent paper is Grinstein. G. and Jayaprakash, C. Phys. Rev. *B25, 523* (1982).
109. See Ref. 68 and references cited therein.
110. Callan, C. G. Phys. Rev. D2, 1541 (1970); Symanzik, K. Commun. Math. Phys. 18. 227 (1970); Symanzik, K. Springer Tracts in Modern Physics 57. 222 (1971).

1 I 111. Lipatov, L. N. Zh. Eksp. Teor. Fiz. 72, 411 (1977); English translation, Soviet Physics-JETP 45, 216 (1977).
Brezin, E. Le Guillou; J, C. and Zinn-Justin, J. Phys. Rev. *D15. 1544,* 1558 (1977).

112. Wegner, F. J., 1974, J. Phys. C7, 2098.
113. *See, e.g.* F. Wegner, in *C.* Domb and *M. S.* Green, eds., *Phase Transitions and Critical Phenomena.* (Academic Press, London, 1976) p. 7.
114. See, e.g., Golner, G. Phys. Rev. *B8,* 3419 (1973); Langer, J. S. and Bar-on, M. Ann Phys. (N.Y.) 78, 421 (1973); Bleher, P. M. and Ya. Ya. G. Sinai. Commun. Math. Phys. 45, 247 (1975).
115. Migdal, A. A. Zh. Eksp. Tear. Fiz. 69, 810, 1457 (1975); English Translations: Soviet Physics-JETP 42, 413, 743 (1975), for a review and more references, see Kadanoff L. P. Revs. Mod. Phys. 49,267 (1977).

116. See, e.g., Niemeijer, Th. and Van Leeuwen. J. M. J. in Domb, C. and Green, M. S. eds., Phase Transitions and Critical Phenomena, Vol. 6 (Academic Press, London, 1976), p. 425.
117. See Ref. 80 and e.g., Riedel, E. K. Physica *106A,* 110 (1981) and references cited therein.
118. Burkhardt. T. W., and J. M. J. Van Leeuwen, 1982, *Real-Space Renormalization* (Springer, Berlin)
119. Kadanoff, I.. P., and A. Houghton, 1975, Phys. Rev. *B11, 377.*
120. See Wilson, K. G. Rev. Mod. Phys. 47, 773 (1975) esp. Sect. IV.
121. See Hilhorst H.J. and Van Leeuwen., J. M. J. Physica *106A,* 301 (1981).
122. Hilhorst. H. J., M. Schick, and J. M. J. Van Lceuwen, 1978. Phys. Rev. Lctt. 40, 1605.
123. I thank Yves Parlange for reminding me of this.
124. See, e.g. Swendsen, R. H. Andelman, D. and Berker, A. N. Phys. Rev. *B24, 6732* (1982): Landau, D. P. and Swendscn R. H. Phys. Rev. Lett. 46, 1437 (1981).
125. Bell, T. L. and Wilson, K. G. Phys., Rev. *B10, 3935 (1974).*
126. See, e.g. Griffiths, R. B. Physica *106A, 59* (1981).
127. See, e.g., Mandelbrot, Benoit B. *The Fractal Geometry of Nature* (Freeman, W. H. San Francisco, 1982);
128. Fowler, M. and Zawadowski, A. Solid State Commun. 9, 471 (1971).
129. Wilson, K. G. Revs. Mod. Phys. 47, 773 (1975); Krishna-Murthy, H. R. Wilkins, J. N. and Wilson, K. G. Phys. Rev. Letters 35, 1101 (1975); Phys. Rev. *B21,* 1044, 1104 (1980).
130. See. e.g. Drcll. S. D. Weinstein, M. and Yankielowicz, S. Phys. Rev. *D17, 1769 (1977);* Jullien. R. Fields. ,J. N. and Doniach. S. Phys. Rev. *Bl6. 4889* (1977); recent references are Hanke, W. and Hirsch, J. E. Phys. Rev. *B25,* 6748 (1982); Penson, K. A. Jullien, R. and Pfeuty, P. Phys. Rev. *B25, 1837* (1982) and references cited therein.
131. Politzcr, H. Phys. Rev. Lett. 30, 1346 (1973); Gross, D. and Wilczek, F. Phys. Rev. Lett. 30, 1343 (1973).
132. Ser. e.g. Altarelli, G. Phys. Repts. *81,* 1 (1982).
133. Wilson, K. G. Phys. Rev. *D3,* 1818 (1971).
134. Polyakov A. M. (unpublished); Wilson, K. G. Phys. Rev. *D10. 2455* (1974).
135. Wegner. F.J. Math. Phys. *12, 2259* (1971).
136. See the review of Bander, M. Phys. Repts. 75, 205 (1981).
137. Swendsen, R. H.. *1982,* in *Real-Space Renormalization,* edited by J.W Burkhardt and J. M. J. Van Leeuwen, (Springer. Berlin).
138. Langacker P., 1981. Phys. Rep. 72, 183.
139. Feigenbaum, M.J.J, Stat. Phys. *19, 25* (1978).
140. Sec. e.g. Eckmann. J. P. Rev. Mod. Phys. 53. 643 (1981): Ott, E. Rev. Mod. Phys. 53. 655 (1981).
141. See e.g. Coppersmith, S. and Fisher, D. Bell Laboratories preprint.
142. See. e.g., Mulliken. R. S. Ann. Rev. Phys. Chem. 29, 1 (1978); Löwdin, P. 0. Adv. Quantum Chem. *12,* 263 (1980); Hirst. D. M. Adv. Chem. Phys. 50,517 (1982); Bartlett, R. J. Ann. Rev. Phys. Chem. 32, 359 (1981): Case, D. A. Ann. Rev. Phys. Chem. 33. 151 (1982).
143. Patashinskii, A. Z.. and V. L. Pokrovskii, 1979. *Fluctuating Theory of Phase Transitions* (Perga-"Ion)
144. Stanley, H. E., 1972, *Introduction to Phase Transitions and* Critical *Phenomena* (Oxford University, London)
145. Widom, B., 1975 in *Fundamental Problems in Statistical Mechanics.* edited by E. C. G. Cohen (North-Holland) Vol. III, p. 1.
146. Fisher, 51. E., 1964, J. Math. Phys. 5, 944.
147. Levy, M., J. C. Le Guillou, and J. Zinn-,Justin, editors, *1980, Phase Transitions* (Plenum, New York).

Books Related to the Renormalization Group

Domb, C. and Green, M. *S.* Eds., *Phase Transitions and Critica/ Phenomena* (Academic Press, London) esp. Vol. 6 (1976);
Pfeuty, P. and Toulouse, G. *Introduction to the Renormalization Group and to Critical Phenomena* (Wiley, Sew York, 1977):

Ma, S.-K. *Modern Theory of Critical Phenomena* (Benjamin/Cummings, Reading, Mass., 1976);
Amit, D. J. *Field Theory, the Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).

Popular Articles (sort of]
Wilson, K. G. Sci. Am. 241, 158 (August, 1979);
Wilson, K. G. Adv. Math 16, 176 (1975).

Review Articles
Greer, S. C. and Moldover, M. R. Ann. Rev. Phys. Chem. 32, 233 (1981): Critical Behavior of Fluids and many references to prior reviews.
Bander, M. ref. 116: Quark Confinement
Zion-Justin, J. (Ref. 67): Precise Computation of critical Exponents from the ε expansion;
Baret, J. F. Prog. Surf. Membrane Sci. 14, 292 (1981);
Cadenhead, D. A. and Danielli, J. F. Eds., (Academic, New York, 1981): A practical area involving phase transitions.
Wallace, D. J. and Zia, R. K. Repts. Progr. Phys. 41, 1 (1978).